ASPECTS OF POSITIVITY IN CONTROL THEORY

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von
Tilman Schanbacher
aus Reutlingen

1986

A S P E C T S O F P O S I T I V I T Y

I N C O N T R O L T H E O R Y

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen

zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von
Tilman Schanbacher
aus Reutlingen

1986

Tag der mündlichen Qualifikation: 30.10.1986

Dekan:	Professor Dr. H. Heyer
1. Berichterstatter:	Professor Dr. R. Nagel
2. Berichterstatter:	Professor Dr. H. H. Schaefer
3. Berichterstatter:	Professor Dr. J. Zabczyk

Meinen Eltern

TABLE OF CONTENTS

Introduction 3

Chapter I Preliminaries 7

1. Sets and Spaces 7

2. Operators 13

3. Semigroups 14

Chapter II Two Examples and Three Questions 18

1. Electrically Heated Oven 18

2. Heat Equation on a Finite Rod with Non-Insulated Ends 22

Chapter III Asymptotic Behaviour of Positive Semigroups .. 25

1. Introduction 25

2. Results 28

3. Applications 35

Chapter IV Controllability 39

1. Introduction 39

2. Controllability without Restrictions on the Controls 40

3. Controllability with Restrictions on the Controls 44

Chapter V Positive Controllability 49
1. Introduction 49
2. General Results 50
3. Finite Dimensional Case 57
4. Infinite Dimensional Case 60
5. Applications 66

Chapter VI Positive Stationary Pairs 72
1. Introduction 72
2. Main Results 73
3. Further Results 79
4. Applications 83

Chapter VII Feedback Controls 85
1. Introduction 85
2. Formulation of the Problems 86
3. A Perturbation Theorem 89
4. Reachability of Growth Bound Zero 93
5. Main Results 97
6. Remarks 110
7. Applications 118

Appendix 123

References 130

INTRODUCTION

In this thesis we are concerned with a linear distributed parameter control system of the form

$$\frac{d}{dt}f(t) = Af(t) + Bu(t) ,$$
$$f(0) = f_o .$$

General questions like controllability and stabilizability of this system have been investigated in the past years intensively by means of functional analysis, especially by using the theory of strongly continuous semigoups; see e.g. the books of Balakrishnan [2], Curtain and Pritchard [5], [6]. Saperstone and Yorke [18], Brammer [4], Korobov and Son [13], [22], and the author [21] considered controllability with the additional requirement that the controls are restricted to a certain subset of the original space, thereby making particular use of the theorems of Krein and Rutman [14] on positive operators.

Being familiar with the theory of positive operators and positive semigroups developed by Schaefer [20] and Nagel [16], we started to look for "aspects of positivity in control theory". On the one hand we expected that for systems described by positive semigroups the (spectral) properties of those semigroups will have some interesting consequences. This will be elaborated in chapter VII. More important however, we felt that up to now there

was paid too less attention to the fact that quite often control of a system can be realized only in one direction. The above mentioned papers on controllability with restrictions on the controls did take this into account, but the goal there was still to reach the whole linear state space or at least a neighborhood of the initial state. However, this is possible only if the system is oscillating in some sense, in particular it is impossible for the large class of systems described by positive semigroups. Our concern is to ask which positive states can be reached for positivity preserving systems if the controls are taken to be positive; and in analogy to the usual definition of controllability we call the system positive controllable if all positive states can be reached.

The chapters I - IV of this thesis are more or less preliminary although they include some new results. The main results are contained in the chapters V - VII. Since an introduction will be given in each of the chapters III - VII, we will now describe the contents of these chapters only briefly, mainly to show the connections between them.

In chapter I we present the definitions and notations needed later, mainly for readers who are not familiar with the abstract theory of Banach lattices and positive operators.

Chapter II is an heuristic introduction into the problems of this thesis. We study two examples in which all the questions arise

which will be considered in the chapters V, VI, VII.

In chapter III some partially new results on the asymptotic behaviour of positive semigroups are presented. The results are needed in chapter V, but they also are quite interesting in themselves, yet not directly related to control theory.

In chapter IV we review the concept of controllability without and with restrictions on the controls, and we will give a counterexample to a conjecture which arose recently. This chapter may be considered as the historical background which led us to the investigation of "positive controllability".

In chapter V we discuss the concept of positive controllability for positive systems, both on finite dimensional and infinite dimensional spaces. We will obtain a neat characterization of positive controllability for finite dimensional spaces, and we will show that positive controllability on infinite dimensional spaces is rather difficult to achieve, contrary to the usual controllability notion in chapter IV.

In chapter VI we consider a subclass of positive states which can be reached easier than arbitrary positive states. Existence of such states is mainly connected with asymptotic stability of the system. This fact connects chapter V to chapter VII, where we consider a (not necessarily positive) asymptotically stable system, and we ask for a feedback control by which we can approx-

imate certain states such that the system remains bounded. For positive systems the states reachable by such a feedback control will be just those discussed in chapter VI. While beeing rather theoretical these results give a valuable insight into the structure of feedback control systems. In addition they are a nice application of many results of the theory of (positive) semigroups.

We will conclude each of the chapters III, V, VI, and VII with practical examples including those presented in chapter II.

This dissertation was done under the guidance of Professor Rainer Nagel. I am also much indebted to Professor Jerzy Zabczyk for his hospitality and encouraging on the occasion of two visits at the Heriot-Watt University in Edinburgh and at the Polish Academie of Sciences in Warsaw. He pointed out many of the problems discussed in this thesis and shared my interest in applying functional analysis to control theory. Furthermore my work was stimulated by Professor A. J. Pritchard during a visit of the Control Theory Centre at the University of Warwick. Finally I thank my collegues in the "Arbeitsgemeinschaft Funktionalanalysis" and especially Dr. Frank Räbiger for many interesting and helpful discussions.

CHAPTER I

PRELIMINARIES

In this chapter we introduce some basic notations and definitions from functional analysis and repeat some standard results from the theory of positive semigroups. We will follow the books of Schaefer [20] and Nagel [16].

I.1 Sets and Spaces

Throughout this thesis we denote by E and U Banach spaces over the same real or complex field, endowed with a norm $\| \ \|$.

By $\mathcal{L}(E)$, $\mathcal{L}(U,E)$ we denote the space of linear continuous (or bounded) operators from E into E, resp. from U into E, endowed with the canonical operator norm.

By $L^p([0,t];U)$ $(1 \leq p \leq \infty)$ we denote the space of all p-integrable functions on $[0,t]$ with values in U, and by $L^p_{loc}(\mathbb{R}_+;U) := \{f: \mathbb{R}_+ \to U : f_{|[0,t]} \in L^p([0,t];U) \text{ for all } t>0\}$ the space of all locally p-integrable functions on $\mathbb{R}_+$ with values in U.

Let Ω be a subset of E or U. Then we denote by

$\mathrm{cl}\Omega$ the closure of Ω,

$\mathrm{co}\Omega$ the smallest convex set containing Ω,

$\mathrm{cone}\Omega$ the smallest cone containing Ω and 0,

$\mathrm{cocone}\Omega$ the smallest convex cone containing Ω and 0,

$\mathrm{span}\Omega$ the smallest linear subspace containing Ω,

$\mathrm{int}\Omega$ the set of all interior points of Ω.

Following [20] we define a (real) Banach lattice E as a Banach space over $\mathbb{R}$ endowed with an order, written as $\leq$, such that $(E,\leq)$ is a lattice and the ordering is compatible with the Banach space structure of E. We are going to elaborate this in more detail.

The axioms of compatibility between the linear structure of E and the order are as follows:

$f \leq g$ implies $f+h \leq g+h$ for all f, g, h in E.

$f \geq 0$ implies $\lambda f \geq 0$ for all f in E and $\lambda \geq 0$.

Any real vector space with an ordering satisfying these two axioms is called an ordered vector space. The axioms imply that the set $E_+ := \{f \in E: f \geq 0\}$ is a convex set and a cone with vertex 0, and therefore called the positive cone of E. It follows that $f \leq g$ if and only if $g - f \in E_+$. The elements $f \in E_+$ are called positive, and we write $f > 0$ if f is positive and different from zero.

An ordered vector space E is called a vector lattice if any two elements f, g in E have a supremum and an infimum, which are denoted by $\sup(f,g)$, resp. $\inf(f,g)$. A vector lattice is called order complete if every majorized subset of E possesses a supremum. For an element f of a vector lattice we write

$$|f| = \sup(f,-f) \quad \text{and call it the absolute value of } f .$$

We call two elements f, g of a vector lattice orthogonal, if $\inf(|f|,|g|) = 0$.

The axiom of compatibility between norm and order requested for a Banach lattice is as follows:

$$|f| \leq |g| \quad \text{implies} \quad \|f\| \leq \|g\| . \tag{1.1}$$

A norm on a vector lattice satisfying this axiom is called a lattice norm. Finally, a (real) Banach lattice is a Banach space E over $\mathbb{R}$ endowed with an ordering $\leq$ such that $(E,\leq)$ is a vector lattice and the norm on E is a lattice norm.

In particular on a Banach lattice we have the formulas

$$|f + g| \leq |f| + |g| \quad \text{and} \quad \||f|\| = \|f\| .$$

A vector subspace F of a vector lattice E is a vector sublattice if

$|f| \in F$ for all $f \in F$, or equivalently if

$\sup(f,g), \inf(f,g) \in F$ for all $f,g \in F$.

A subset G of a vector lattice is called solid if $f \in G$, $|g| \leq |f|$ implies $g \in G$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an ideal. Ideals are automatically vector sublattices since $|\sup(f,g)| \leq |f| + |g|$. A band in a vector lattice E is an ideal which contains arbitrary suprema, more exactly: G is a band if G is an ideal in E and if $\sup M$ is contained in G whenever M is contained in G and $\sup M$ exists in E. Since the notions of sublattice, ideal, band are invariant under the formation of arbitrary intersections there exists, for any subset G of E, a uniquely determined smallest sublattice (ideal, band) of E containing G. It will be called the sublattice (ideal, band) generated by G.

Two ideals I, J of a vector lattice E are called orthogonal if any two elements $f \in I$, $g \in J$ are orthogonal. It is immediate that two ideals are orthogonal if and only if they have trivial intersection $\{0\}$. An ideal I which has a complementary orthogonal ideal J, i.e. $E = I \oplus J$, is a band and it is called a projection band ([20, II.2.7, 2.8]). The projection of E onto a projection band I is called the band projection belonging to I. An ideal I is called proper if I is different from E and from $\{0\}$.

Typical ideals are all sets of the form

$$I_f := \{g \in E : |g| \leq \lambda f,\ \lambda \geq 0\} \quad \text{for some } f \in E_+ \text{, and}$$
$$J_{f'} := \{f \in E : \langle f', |f| \rangle = 0\} \quad \text{for some } 0 \leq f' \in E' .$$

An element f in the positive cone of E is called a quasi-interior point of E_+ if the ideal I_f generated by f is dense in E.

A linear form $f' \in E'$ is called

positive (notation: $f' \geq 0$) if $\langle f', f \rangle \geq 0$ for all $f \geq 0$,
strictly positive ($f' >> 0$) if $\langle f', f \rangle > 0$ for all $f > 0$.

If E is a Banach lattice, then the dual space E' with the natural ordering (given by $f' \leq g'$ if and only if $\langle f', f \rangle \leq \langle g', f \rangle$ for all $f \in E_+$) and the natural norm is an order complete Banach lattice ([20, II.5.5]).

A Banach lattice is called an AL-space if the norm has the property that $\|f\| + \|g\| = \|f+g\|$ for all $f, g \geq 0$.

All finite dimensional Banach lattices of the same dimension are isomorphic ([20, II.3.9 Cor. 1 and II.5.1]), in particular isomorphic to an AL-space.

A complex Banach lattice E is the complexification of a real Banach lattice $E_{\mathbb{R}}$ in the sense that

$$E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}} ,$$

i.e. $E = E_{\mathbb{R}} \times iE_{\mathbb{R}}$ with scalar multiplication $(\alpha+i\beta)(f,g) = (\alpha f-\beta g,\beta f+\alpha g)$. $E_{\mathbb{R}}$ is called the real part of E . The absolute value of an element $h = f + ig \in E$ is defined by

$$|h| = \sup\{ \cos\theta\cdot f + \sin\theta\cdot g : 0 \leq \theta \leq 2\pi \} \in E_{\mathbb{R}} ,$$

and the norm still satisfies (1.1).

Since the absolute value exists for all $f \in E$, the definition of an ideal can be extended unchanged to the complex situation. Most of the terms introduced above for real Banach lattices can be extended to the complex situation under retention (mutatis mutandis) of the corresponding results valid in the real case by either using the complex absolute value or else, if the formation of suprema or infima is involved, by relating them to real parts. For example $f \in E$ is called positive if $f = |f|$ which means that f is a positive element of $E_{\mathbb{R}}$.

In the following a Banach lattice denotes either a real or a complex Banach lattice.

Particular Banach lattices are the following spaces:
$\mathbb{R}^n$, $n \in \mathbb{N}$ with the coordinatewise ordering;

the function spaces $C_o(Y)$ (Y locally compact) and $L^p(\Omega)$ (Ω a measure space, $1 \leq p \leq \infty$) with the ordering $f \leq g$ for $f,g \in C_o(Y)$, resp. $L^p(\Omega)$, if $f(x) \leq g(x)$ for all $x \in Y$, resp. for almost all $x \in \Omega$.

I.2 Operators

Let E and U be Banach spaces, $T \in L(E)$ and $S \in L(U,E)$.

We denote the kernel of S by

$$\mathrm{Ker}S := \{u \in U : Su = 0\} .$$

The spectrum $\sigma(T)$ of T is defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda\text{-}T \text{ has no inverse in } L(E)\} ,$$

the point spectrum as

$$p\sigma(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\} ,$$

the resolvent set as

$$\rho(T) := \mathbb{C} \setminus \sigma(T) ,$$

and the spectral radius of T as

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} .$$

It is known that $r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}$.

For all $\lambda \in \rho(T)$ we can define the resolvent

$$R(\lambda,T) := (\lambda Id - T)^{-1} ,$$

where

Id denotes the identity on E.

We call an eigenvalue $\lambda \in p\sigma(T)$ <u>simple</u> if its eigenspace

$\mathrm{Ker}(\lambda-T)$ is one dimensional.

Let $\lambda_o \in \sigma(T)$ be a pole of order m of the resolvent of T, i.e. λ_o is an isolated singular point in $\sigma(T)$ and the Laurent expansion of $R(\lambda,T)$ in a neighborhood of λ_o is given by $\sum_{n=-m}^{\infty} T_n(\lambda-\lambda_o)^n$, $T_{-m} \neq 0$. Then $P := T_{-1}$ is called the residue at λ_o, and P is a linear continuous projection onto $\mathrm{Ker}(\lambda_o-T)^m$ with $\sigma(T_{|PE}) = \{\lambda_o\}$.

We define $r_{ess}(T)$ (the essential spectral radius of T) as the infimum of all $r \in \mathbb{R}_+$ such that every $\lambda \in \sigma(T)$, $|\lambda|>r$ is a pole of the resolvent of T with residue of finite rank.

Let now E, U be Banach lattices.

Then S is called positive (notation: $S \geq 0$) if $Su \geq 0$ for $0 \leq u \in U$.

S is called strictly positive ($S >> 0$) if $Su > 0$ for all $0 < u \in U$.

$T \geq 0$ is called irreducible if no proper ideal in E is invariant under T.

I.3 Semigroups

We consider a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ of linear continuous operators on a Banach space E, i.e.

$$T(t+s) = T(t)\cdot T(s)$$

$$T(0) = Id$$

$$\lim_{t\to 0} T(t)f = f \quad \text{for all } f \in E.$$

In the following we simply call T a semigroup on E.
Let $(A,D(A))$ be its generator, i.e.

$$D(A) := \{f \in E : \lim_{t\to 0} \frac{T(t)f - f}{t} \text{ exists}\},$$

$$Af := \lim_{t\to 0} \frac{T(t)f - f}{t} .$$

Then $(A,D(A))$ is a closed operator on E which determines T uniquely. In general, A is unbounded but $\sigma(A)$, $p\sigma(A)$, $\rho(A)$, $R(\lambda,A)$ and poles of the resolvent of A can be defined as in the previous section for bounded operators;
$s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ is called the spectral bound of A (or of T), and $\omega(A) := \omega(T) :=$
$\inf\{\omega \in \mathbb{R} :$ There exists M_ω such that $\|T(t)\| \leq M_\omega e^{-\omega t}$ for all $t \geq 0\}$ the growth bound of A (or of T).
Then $-\infty \leq s(A) \leq \omega(A) < +\infty$ and $r(T(t)) = e^{t\omega(A)}$ for $t \geq 0$ holds ([16, A-I,1.11, A-III,1.1]).
For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega(A)$ the resolvent $R(\lambda,A)$ is given by ([16, A-I,1.11])

$$R(\lambda,A)f = \int_0^\infty e^{-\lambda t} T(t) f dt . \qquad (1.2)$$

A real eigenvalue $\mu \in \sigma(A)$ is called strictly dominant if $\mu > \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\setminus\{\mu\}\}$.
By [16, A-III,3.7] $\omega_{ess}(A) := \omega_{ess}(T) \in [-\infty,\infty)$ is well defined by

$$r_{ess}(T(t)) = \exp(t\cdot\omega_{ess}(T)) \quad \text{for all } t \geq 0$$

and it is called the essential growth bound of A (or of T).

The semigroup T is called uniformly continuous if $t \to T(t)$ is a continuous map from $\mathbb{R}_+$ into $L(E)$, or equivalently if $A \in L(E)$; in this case we have $T(t) = \exp(tA) := \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n$ ([16, A-I,2.1]).

T is called holomorphic if for some $\alpha > 0$ the map $t \to T(t)$ is the restriction of a holomorphic function $T : S(\alpha) := \{r \cdot e^{i\vartheta} : r > 0, \vartheta \in (-\alpha,\alpha)\} \to L(E)$ satisfying $T(z+z') = T(z) \cdot T(z')$ for $z,z' \in S(\alpha)$ and $\lim_{z \to 0, z \in S(\alpha)} T(z)f = f$ for every $f \in E$.

T is called differentiable if $t \to T(t)f$ is a differentiable map from $(0,\infty)$ into E for every $f \in E$, or equivalently if $T(t)f \in D(A)$ for every $f \in E$ and $t > 0$ ([16, A-II,1.16]).

T is called eventually norm continuous if there exists t_o such that $t \to T(t)$ is continuous from (t_o,∞) into $L(E)$.

T is called eventually compact if there existst t_o such that $T(t_o)$ is a compact operator on E.

The following implications hold: Uniformly continuous $\to$ holomorphic $\to$ differentiable $\to$ eventually norm continuous; eventually compact $\to$ eventually norm continuous ([16, A-II,1, p.41]). An eventually norm continuous semigroup is eventually compact if the resolvent $R(\lambda,A)$ is compact for some $\lambda \in \rho(A)$ ([16, A-II,1.24]). Furthermore "$s(A) = \omega(A)$" for every eventually norm continuous semigroup ([16, A-III,6.6]).

The semigroup T is called bounded if there exists $M > 0$ such that $\|T(t)\| \leq M$ for every $t \geq 0$ (this implies $\omega(A) \leq 0$); T is called a contraction semigroup if M can be chosen equal to 1.

A special class of bounded semigroups are the following semigroups.

T is called weakly stable if

$\langle f', T(t)f \rangle \to 0$ as $t \to \infty$ for every $f \in E$ and $f' \in E'$,

strongly stable if $\|T(t)f\| \to 0$ as $t \to \infty$ for every $f \in E$,

uniformly exponentially stable if $\omega(A) < 0$.

On finite dimensional spaces these three stability notions coincide (and we simply call the semigroup stable), but they do not coincide on infinite dimensional spaces ([16, A-IV,1]). Of course, uniformly exponential stability implies strong stability, and strong stability implies weak stability.

Let now E be a Banach lattice. Then the semigroup T is called positive (notation: $T > 0$) if $T(t)$ is a positive operator for every $t \geq 0$. A positive semigroup T is called irreducible if there is no proper T-invariant ideal (the single operator $T(t)$ itself may not be irreducible!). Here, an ideal I in E is called T-invariant if it is invariant under all $T(t)$, $t \geq 0$.

For a positive semigroup on a Banach lattice formula (1.2) holds for all λ with $\mathrm{Re}\lambda > s(A)$ [16, C-III,1.2].

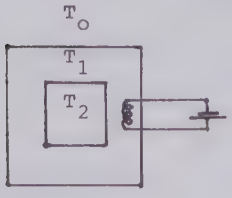
CHAPTER II

TWO EXAMPLES AND THREE QUESTIONS

As heuristic motivation for this paper, we present two practical examples of linear control systems. The first is a finite dimensional system, while in the second infinite dimensional spaces occur. In both the solution of the system remains in the positive cone of a Banach lattice, and we can pose three questions which will be discussed in the chapters V, VI and VII respectively. We will come back to these two examples in each of these chapters.

II.1 Electrically Heated Oven

We consider a simple model of an electically heated oven (see [3, Ex. 1.2], which consists of a jacket, an interior part, and a coil which directly heats up the jacket and indirectly the interior part by means of radiation of the jacket.



Let T_o denote the outside temperature, $T_1(t)$ the temperature of the jacket at time t, and $T_2(t)$ the temperature of the interior part at time t.

If we set $f(t) = \begin{pmatrix} T_1(t) - T_o \\ T_2(t) - T_o \end{pmatrix}$ then the system is described by

$$\frac{d}{dt}f(t) = \begin{pmatrix} -\alpha & \beta \\ \gamma & -\delta \end{pmatrix} f(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \,, \qquad f(0) = f_o \,, \tag{2.1}$$

where $u(t)$ measures the heat input at time t and is always non-negative (since it is proportional to the square of the voltage in the coil); $\alpha, \beta, \gamma, \delta$ are non-negative constants and f_o is assumed to be non-negative ($f_o \in \mathbb{R}^2_+$). This means that the differences $T_1(t)-T_o$ and $T_2(t)-T_o$ at time $t=0$ are non-negative. It seems plausible that then $f(t)$ is non-negative for all times. Indeed, if we denote by A the matrix $\begin{pmatrix} -\alpha & \beta \\ \gamma & -\delta \end{pmatrix}$ and by B the matrix $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then by [6, 2.23] for every $u \in L^1_{loc}(\mathbb{R}_+)$ a mild solution of (2.1) is given by

$$f(t) = T(t)f_o + \int_0^t T(t-s)Bu(s)ds \,, \tag{2.2}$$

where $T = (T(t))_{t \geq 0}$ is the stongly continuous semigroup generated by A. T is positive by [16, B-II,1.4 (b)]. From (2.2) we see that $f(t) \in \mathbb{R}^2_+$ for all $t \geq 0$.

The first question is now whether each $f_1 \in \mathbb{R}^2_+$ can be approximately reached from the origin ($f_o = 0$) by means of a suitable non-negative control u.

It seems possible that by means of a high input one can reach at least for a short period of time an arbitrary high jacket tem-

perature while simultaneously the interior temperature is low. However the opposite, low jacket temperature and arbitrary high interior temperature seems to be impossible. Therefore we expect a negative answer to this question, what is not so disappointing because not all states are desirable to reach.

For our second question we consider a special class of states f_1, namely those belonging to a positive stationary pair (f_1,u_1). This means that f_1 is a constant positive solution of (2.1) for constant non-negative input u_1 and initial state $f_o = f_1$, i.e. $Af_1 + Bu_1 = 0$, $0 \neq f_1 \in \mathbb{R}^2_+$, $u_1 \in \mathbb{R}_+$. If we can reach these states f_1 then we can held them constant by further input u_1. If the system is stable and if we can reach them only approximately then it should also be possible to hold them approximately constant by further input u_1.

Therefore we ask as our second question: Can we (approximately) reach all states f_1 belonging to a positive stationary pair starting from any $f_o \in \mathbb{R}^2_+$. The question makes sense only if there exists a positive stationary pair, and moreover stability of the system is desirable as just mentioned.

In the concrete example and if $\gamma + \delta \neq 0$ we can show by a simple calculation the following equivalences:

A positive stationary pair exists if and only if A has only eigenvalues with non-positive real part,

and

A positive stationary pair (f_1,u_1) with $u_1 \neq 0$ exists if and only if A has only eigenvalues with negative real part.

In practice (see [3, Ex. 1.2]) the parameters $\alpha, \beta, \gamma, \delta$ will be (strictly) positive and also such that A is stable.

The third question is a modification of the second one for feedback-controls arising often in practical situations, i.e. , we consider controls of the form $u(t) = Cf(t)$, where C is a linear operator from $\mathbb{R}^2$ into $\mathbb{R}$. Given a state f_1 belonging to a positive stationary pair we seek in our third question a feedback operator C such that we can approximately reach f_1 or a positive scalar multiple starting from any $0 \neq f_o \in \mathbb{R}^2_+$.

It is clear that we cannot (approximately) reach f_1 itself from any $0 \neq f_o \in \mathbb{R}^2_+$ since for feedback controls the final state depends linearly on the initial state. But it suffices to reach some positive scalar multiple of f_1. In plain words this means that for some $\lambda \in [0,\infty]$ we can design a feedback control such that starting from any initial state we can approximately reach a state for which the ratio of temperatures $(T_1-T_o)/(T_2-T_o)$ is equal to λ.

II.2 Heat Equation on a Finite Rod with Non-Insulated Ends

We consider the heat equation on a rod of length one with non-insulated ends which is given by

$$\begin{aligned} \frac{\partial}{\partial t}\chi(t,x) &= \frac{\partial^2}{\partial x^2}\chi(t,x) \ , \quad 0\le x\le 1 \ , \ t\ge 0 \ , \\ \chi(t,0) &= \chi(t,1) = 0 \ , \\ \chi(0,x) &= \chi_o(x) \ , \end{aligned} \tag{2.3}$$

where χ_o is a given non-negative function on $[0,1]$.

Again we wish to control the system by a non-negative one dimensional input, i.e. by a function $\psi(x)\cdot u(t)$, where $\psi(x)\ge 0$ for $0\le x\le 1$, $u(t)\ge 0$ for $t\ge 0$. As in the first example, we can interpret this control as an electrical heating input which for all times is proportional to a given heat distribution ψ. Then we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\chi(t,x) &= \frac{\partial^2}{\partial x^2}\chi(t,x) + \psi(x)\cdot u(t), \quad 0\le x\le 1, \ t\ge 0 \ , \\ \chi(t,0) &= \chi(t,1) = 0 \ , \\ \chi(0,x) &= \chi_o(x), \end{aligned} \tag{2.4}$$

and again it seems plausible that χ remains non-negative for all times.

We state this problem as an abstract control problem on the space $E := L^2[0,1]$. Assume $\psi\in E$ and let $U = \mathbb{R}$, $B\in\mathcal{L}(U,E)$, $B\alpha := \alpha\cdot\psi$, $f_o := \chi_o$, and let $A = \frac{d^2}{dx^2}$ be the operator on E with domain $D(A) = \{f\in E : f''\in E, f(0)=f(1)=0\}$. It is known ([6, 3.9],

[16, C-II,1.5(b)]) that $(A,D(A))$ generates on E the positive semigroup $T = (T(t))_{t\geq 0}$ which is given by

$$T(t)f = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle f,e_n\rangle e_n \ , \tag{2.5}$$

where $f\in E$, $e_n(x) = \sqrt{2}\cdot\sin(n\pi x)$ for $0\leq x\leq 1$, $\langle f,e_n\rangle := \int_0^1 f(x)e_n(x)dx$. Then (2.4) can be written as (set $f(t) := \chi(t,\cdot) \in E$):

$$\begin{aligned} &\frac{d}{dt}f(t) = Af(t) + Bu(t) \ , \\ &f(t) \in D(A) \ , \\ &f(0) = f_o, \end{aligned} \tag{2.6}$$

and for $u\in L^1_{loc}(\mathbb{R}_+)$ a mild solution is again given by (2.2).

We observe that the semigroup T is uniformly exponentially stable:

$$\begin{aligned} \|T(t)f\| &\leq \Big(\sum_{n=1}^{\infty} e^{-2n^2\pi^2 t}\langle f,e_n\rangle^2\Big)^{1/2} \\ &\leq e^{-\pi^2 t}\Big(\sum_{n=1}^{\infty}\langle f,e_n\rangle^2\Big)^{1/2} = e^{-\pi^2 t}\|f\|; \end{aligned}$$

and we claim that a stationary pair exists.

Proof: Let

$$f(x) := -\int_0^x\int_0^y \psi(z)dzdy + x\int_0^1\int_0^y \psi(z)dzdy \ ,$$

where ψ as above, then $f\in D(A)$, $Af(x) = -\psi(x)$.

Assume that f is not contained in the positive cone of E. Since $f(0) = f(1) = 0$ this implies that f has a negative minimum in some $x_o\in(0,1)$. Then

$$0 = f'(x_o) = -\int_0^{x_o} \psi(z)dz + \int_0^1 \int_0^y \psi(z)dzdy ,$$

and since $\psi \geq 0$ we obtain

$$f'(x) = -\int_0^x \psi(z)dz + \int_0^1 \int_0^y \psi(z)dzdy \geq f'(x_o) = 0 \text{ for } 0 \leq x \leq x_o ,$$

$$f(x_o) = f(0) + \int_0^{x_o} f'(x)dx = 0 + \int_0^{x_o} f'(x)dx \geq 0,$$

which is a contradiction. Hence $f \geq 0$, $Af + B1 = 0$, i.e. $(f,1)$ is a stationary pair.

□

Therefore it is interesting to ask the same three questions as in the first section.

CHAPTER III

ASYMPTOTIC BEHAVIOUR OF POSITIVE SEMIGROUPS

III.1 Introduction

Let E be a Banach lattice, $T \in \mathcal{L}(E)$ positive and $\mathcal{T} = (T(t))_{t\geq 0}$ a positive semigroup on E with generator $(A,D(A))$. In this chapter we will consider the asymptotic behaviour of $\frac{T^n f}{\|T^n f\|}$ as $n\to\infty$ and of $\frac{T(t) f}{\|T(t) f\|}$ as $t\to\infty$ for $0<f\in E$.

The Perron-Frobenius-theory of positive operators (see e.g. [20, chap. I and V]) and its generalization to positive semigroups by Greiner [9] is extensively studied and yielded results on the asymptotic behaviour of T^n as $n\to\infty$ resp. $T(t)$ as $t\to\infty$. More particular, we know that the spectral radius $r(T)$ of T and the spectral bound $s(A)$ of $\mathcal{T}$ are always elements of the spectra $\sigma(T)$ resp. $\sigma(A)$ (unless $s(A) = -\infty$). If $r(T)$ is a pole of order 1 of the resolvent of T and if $|\lambda|<r(T)$ for all other $\lambda\in\sigma(T)$ then $(\frac{T}{r(T)})^n$ tends as $n\to\infty$ to a projection P, and P is strictly positive if T is an irreducible operator. If $s(A)$ is a first order pole of the resolvent of A then it is also known that in many instances $e^{-s(A)t}T(t)$ tends as $t\to\infty$ to a projection P which is strictly positive if $\mathcal{T}$ is irreducible. In both cases convergence of the considered terms obviously implies convergence of $T^n f\cdot(\|T^n f\|)^{-1}$ as $n\to\infty$ for every $0<f\in E\setminus\bigcup_{n\in\mathbb{N}}\mathrm{Ker}(T^n)$, resp. of

$\frac{T(t)f}{\|T(t)f\|}$ as $t\to\infty$ for every $0<f\in E\setminus U_{t\geq 0}\mathrm{Ker}(T(t))$. As the following example shows the converse of these implications is not true.

Example 3.1

(a) Let $E = \mathbb{R}^2$, $T = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, then $T^n = \begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$, $r(T) = 1$ and the limit of $(\frac{T}{r(T)})^n$ as $n\to\infty$ does not exist (1 is a second order pole of the resolvent of T), but

$$\frac{T^n\begin{pmatrix}x\\ y\end{pmatrix}}{\left\|T^n\begin{pmatrix}x\\ y\end{pmatrix}\right\|} = \frac{\begin{pmatrix}x\\ nx+y\end{pmatrix}}{(x^2+n^2x^2+2nxy+y^2)^{1/2}} \to \begin{pmatrix}0\\ 1\end{pmatrix} \quad \text{as } n\to\infty \text{ for } 0<\begin{pmatrix}x\\ y\end{pmatrix}\in E.$$

(b) Let $E = \mathbb{R}^2$, $A = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Then $\exp(tA) = \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$, $s(A) = 0$ and $\lim_{t\to\infty}\exp(-s(A)t)\cdot\exp(tA)$ does not exist, but

$$\frac{\exp(tA)\begin{pmatrix}x\\ y\end{pmatrix}}{\left\|\exp(tA)\begin{pmatrix}x\\ y\end{pmatrix}\right\|} = \frac{\begin{pmatrix}x\\ tx+y\end{pmatrix}}{(x^2+t^2x^2+2txy+y^2)^{1/2}} \to \begin{pmatrix}0\\ 1\end{pmatrix} \quad \text{as } t\to\infty \text{ for } 0<\begin{pmatrix}x\\ y\end{pmatrix}\in E.$$

In fact we will see in 3.5 that for positive semigroups on finite dimensional Banach lattices $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ always exists for $0 < f \in E$. This is no longer true on infinite dimensional spaces (Prop. 3.4), but we will find in 3.5, 3.7, 3.9 and 3.10 sufficient conditions under which the limit exists for all $0 < f \in E$.

Moreover, even on finite dimensional spaces $\lim_{n\to\infty}\frac{T^nf}{\|T^nf\|}$ does in general not exist for $0<f\in E\setminus U_{n\in\mathbb{N}}\mathrm{Ker}(T^n)$ as the following example shows.

Example 3.2

Let $E = \mathbb{R}^2$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\lim_{n\to\infty} \frac{T^n f}{\|T^n f\|} = \lim_{n\to\infty} \frac{T^n f}{\|f\|}$ exists only if f is a scalar multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Results on the asymptotic behaviour of $\frac{T^n f}{\|T^n f\|}$ were obtained in a more general form in [7]. They can be summarized for linear operators in the following theorem.

Theorem 3.3 ([7])

Let E be a Banach lattice and $T \in \mathcal{L}(E)$ be a positive operator.

The equation $Tf = \lambda f$, $f \in E_+$, $\|f\| = 1$, $\lambda > 0$ has a unique solution (f^*, λ^*) and $\frac{T^n f}{\|T^n f\|}$ tends as $n \to \infty$ to f^* for all $0 < f \in E$ if one of the following conditions is fulfilled:

(i) $E = \mathbb{R}^n$; $f < g$ implies $Tf << Tg$ (i.e. all coordinates of $Tg - Tf$ are positive).

(ii) There exists $e \in E_+$ with $\|e\| = 1$ and $\alpha, \beta > 0$ such that $\alpha e \leq Tf \leq \beta e$ for all $f \in E_+$ with $\|f\| = 1$.

We remark that condition (i) implies irreducibility of the operator T; (ii) implies that $T : E \to (I_e, p_e)$ is continuous and that $T_{|I_e} : (I_e, p_e) \to (I_e, p_e)$ is an irreducible operator, where I_e denotes the ideal generated by e as defined in I.1, and p_e the canonical gauge functional which makes I_e into a Banach lattice. In particular, T is irreducible if e is an order unit in E (see [20, II.2]).

However, as Example 3.2 shows irreducibility of T is not enough to obtain existence of $\lim_{n\to\infty} \frac{T^n f}{\|T^n f\|}$.

III.2 Results

The following proposition gives a necessary condition for the existence of $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$, and it shows that this limit does not exist for any $f\in E_+$ if A has void point spectrum. An example of a positive semigroup with $p\sigma(A) = \emptyset$ is the translation semigroup $T = (T(t))_{t\geq 0}$ on $L^p(\mathbb{R}_+)$, $1\leq p<\infty$ given by $T(t)f(x) = \begin{cases} f(x-t) & \text{for } 0\leq t\leq x \\ 0 & \text{for } x<t \end{cases}$ with generator (A,D(A)) given by $D(A) = \{f\in L^p(\mathbb{R}_+) : f \text{ is absolutely continuous, } f'\in L^p(\mathbb{R}_+), f(0)=0\}$, $Af = -f'$. An example of a positive semigroup with bounded generator A and $p\sigma(A) = \emptyset$ is the semigroup generated by the right-shift operator on l^p, $1\leq p\leq\infty$: $A(x_1,x_2, \dots) = (0,x_1,x_2, \dots)$.

Proposition 3.4

Let $T = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space E with generator (A,D(A)). If $g := \lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ exists for some $f\neq 0$ then g is an eigenvector to a real eigenvalue of A.

Proof: For all $s\geq 0$ we have by continuity of T

$$\|T(s)g\| = \|T(s)\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}\| = \lim_{t\to\infty}\frac{\|T(s+t)f\|}{\|T(t)f\|}.$$

Hence

$$T(s)g = \lim_{t\to\infty}\frac{T(s+t)f}{\|T(t)f\|} = \lim_{t\to\infty}\frac{T(s+t)f}{\|T(s+t)f\|}\cdot\frac{\|T(s+t)f\|}{\|T(t)f\|} = g\cdot\|T(s)g\|.$$

Thus $T(s)g \in \text{span}(g)$ for every $s\geq 0$ and therefore $g \in D(A)$, $Ag \in \text{span}(g)$.

□

In following three theorems 3.5, 3.7 and 3.10 existence of $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ for all $f>0$ can be shown. In particular this limit exists by Theorem 3.5 on all finite dimensional Banach lattices. We remark that under the assumptions of these theorems the limit does not exist in general for arbitrary $f\in E$: If λ is a non-real eigenvalue of A, then $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ does not exist for a corresponding eigenvector f (resp. for its real part if E is a real Banach lattice).

Theorem 3.5

Let $T = (T(t))_{t\geq 0}$ be a positive eventually compact semigroup on a Banach lattice E with generator $(A,D(A))$.
Then $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ exists for every $f>0$ if and only if there exists no T-invariant ideal $I \neq \{0\}$ in E such that $\sigma(A_{|I}) = \emptyset$.

Proof: The necessity of the condition follows immediately from Proposition 3.4.
To show sufficiency we first remark that by I.3 every eventually compact semigroup is eventually norm continuous. By assumption and [16, C-III,1.1] we know that for any T-invariant ideal $I \neq \{0\}$ $\omega(A_{|I}) = s(A_{|I}) \in \sigma(A_{|I})$. By [16, A-III,6.5] and since $T = (T(t))_{t\geq 0}$ is eventually compact $s(A_{|I})$ is a pole of the resolvent of $A_{|I}$ with residue P_I. By [16, C-III,2.13] and since T is eventually norm continuous $s(A_{|I})$ is a strictly dominant eigenvalue of $A_{|I}$. If $s(A_{|I})$ is a pole of order k, then by [16, A-III,3.6, C-III,1.1] the $-k^{th}$ coefficient Q_I of the

Laurent expansion of $R(\cdot, A_{|I})$ is a positive operator, and $Q_I = (A-s(A_{|I}))^{k-1}P_I$.

We now show that for every $f>0$ there exists a T-invariant ideal I containing f such that $Q_I f \neq 0$.

Suppose to the contrary that there exists $f>0$ such that $Q_I f = 0$ for every T-invariant ideal I.

Since $Q_I > 0$ commutes with $T(t)$ we obtain a sequence of T-invariant ideals

$$I_1 := E,\ I_{i+1} := \{g \in I_i : Q_{I_i}|g| = 0\}$$

such that $f \in I_i$ for all i .

Define $Q_i := Q_{I_i}$ and $s_i := s(A_{I_i})$. By [16, C-III,2.11, B-III,2.8] we know that $s_{i+1} \leq s_i$ for all i, and to every $i \geq 1$ there exists $j \geq 1$ such that $s_{i+j} < s_i$ (more precisely: $s_{i+k} < s_i$ if s_i is a pole of order k of the resolvent of $A_{|I_i}$).

By [16, A-III,6.6] and since $T(t)$ is compact for some $t>0$ we know that $\sigma(A)$ consists of isolated points. Therefore $s_i \to -\infty$ as $i \to \infty$. Hence $I := \bigcap_{i=1}^{\infty} I_i$ is a T-invariant ideal containing f, and $s(A_{|I}) = -\infty$, i.e. $\sigma(A_{|I}) = \emptyset$, which contradicts our assumption.

Thus we have shown that $Q_I f \neq 0$ for some T-invariant ideal I containing f.

Without loss of generality we may assume that $I = E$ (otherwise restrict T to I) and that $s(A) = 0$. Define $P := P_I$, $Q := Q_I$ and let $s(A) = 0$ be a pole of order k of the resolvent of A. Then $Qf = A^{k-1}Pf \neq 0$,

$$T(t)Pf = \sum_{n=0}^{k-1} \frac{1}{n!}(tA)^n Pf \ ,$$

and $Pf, APf, \ldots, A^{k-1}Pf$ are linearly independent elements in E.

Hence $PT(t)f = T(t)Pf \neq 0$ and $T(t)f \neq 0$ for all $t \geq 0$. Furthermore $\omega(A_{|(Id-P)E}) < 0$ since 0 is a strictly dominant eigenvalue. Apply now the subsequent Lemma 3.6 to

$$f(t) := \frac{1}{k-1!}(tA)^{k-1}Pf \text{ and } g(t) := T(t)(Id-P)f + \sum_{n=0}^{k-2} \frac{1}{n!}(tA)^n Pf$$

to obtain

$$\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|} = \frac{A^{k-1}Pf}{\|A^{k-1}Pf\|} = \frac{Qf}{\|Qf\|} .$$

□

It remains to show the following lemma.

Lemma 3.6

Let E be a Banach space, $f(t), g(t) \in E$, $f(t) \neq 0$, $f(t)+g(t) \neq 0$ for all $t>0$ and $\frac{\|g(t)\|}{\|f(t)\|} \to 0$ as $t\to\infty$; then $\frac{g(t)+f(t)}{\|g(t)+f(t)\|} - \frac{f(t)}{\|f(t)\|} \to 0$ as $t\to\infty$.

Proof: By assumption we have as $t\to\infty$

$$\frac{g(t)}{\|g(t)+f(t)\|} = \frac{g(t)}{\|f(t)\|} \cdot \left\| \frac{g(t)}{\|f(t)\|} + \frac{f(t)}{\|f(t)\|} \right\|^{-1} \to 0 \cdot 1 = 0 , \quad (3.1)$$

hence

$$\left\| \frac{f(t)}{\|g(t)+f(t)\|} - \frac{f(t)}{\|f(t)\|} \right\| = \left\| \frac{f(t)}{\|f(t)\|} \left(\frac{\|f(t)\|}{\|g(t)+f(t)\|} - 1 \right) \right\| = \quad (3.2)$$

$$\left| \frac{\|f(t)\|}{\|g(t)+f(t)\|} - \frac{\|g(t)+f(t)\|}{\|g(t)+f(t)\|} \right| \leq \frac{\|g(t)\|}{\|g(t)+f(t)\|} \to 0.$$

(3.1) and (3.2) yield the claim.

□

Theorem 3.7

Let $T = (T(t))_{t\geq 0}$ be a positive strongly continuous semigroup on a Banach lattice E with generator (A,D(A)). Let the spectral bound s(A) of A be equal to the growth bound $\omega(A)$, let s(A) be a pole of first order of the resolvent of A with strictly positive spectral projection P and let $\omega(A_{|(Id-P)E}) < \omega(A)$. Then $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ exists for every $0<f\in E$.

Remark 3.8

(a) We mention that in the situation of 3.7 $\lim_{t\to\infty} e^{-s(A)t}T(t)$ exists and is equal to P.

(b) In view of Theorem 3.3 we remark that $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ is equal to $\frac{Pf}{\|Pf\|}$ and that the set $\{\frac{Pf}{\|Pf\|} : f>0\}$ is exactly the set of the solutions of the equation $Af = \lambda f$, $0<f\in E$, $\lambda\in\mathbb{C}$, $\|f\|=1$. The solution is unique if P has rank 1.

Proof: PE is the eigenspace to the eigenvalue s(A) of A; furthermore for $f \neq 0$ satisfying $Af = \lambda f$ for some $f\in\mathbb{C}$ we have either $\lambda \neq s(A)$ and therefore $f\in(Id-P)E$, $Pf = 0$, $f\notin E_+$ (since $P>>0$) or $\lambda = s(A)$ and therefore $f = Pf$.

Proof of 3.7: The proof of the theorem follows immediately from 3.8 (a) and the remarks in the first section of this chapter before 3.1.

□

Corollary 3.9

Let $T = (T(t))_{t>0}$ be an irreducible positive eventually norm continuous semigroup on a Banach lattice E with generator (A,D(A)). If s(A) is a pole of the resolvent of A then $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ exists for all $0<f\in E$.

Proof: Without loss of generality we can assume that s(A)=0. Since T is irreducible we know by [16, C-III, 3.12] that $\sigma(A)\cap i\mathbb{R}$ is an additive group and consists only of first order poles of the resolvent of A with one dimensional eigenspaces. Let P be the spectral projection to 0. Since T is eventually norm continuous $\sigma(A)\cap(\lambda+i\mathbb{R})$ is bounded for all $\lambda\in\mathbb{R}$ by [16, A-II, 1.20], hence $\sigma(A)\cap i\mathbb{R} = \{0\}$ and $s(A_{|(Id-P)E}) < s(A) = 0$. Furthermore eventually norm continuity implies by [16, A-III, 6.6] that the spectral and growth bound coincide, in particular $T(t) \to P$ as $t\to\infty$ and by [20, III.8.5] irreducibility of T implies that P is strictly positive. By 3.7 the assertion follows.

□

Theorem 3.10

Let $T = (T(t))_{t\geq 0}$ be a positive semigroup on an AL-space E with bounded positive generator $A \in L(E)$. If for every $0<f\in E$ either $A^n f = 0$ for some $n\in\mathbb{N}$ or $\lim_{n\to\infty} \frac{A^n f}{\|A^n f\|}$ exists, then $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ exists for every $0<f\in E$.

Proof: If $A^n f = 0$ for some n and $A^m f \neq 0$ for $m<n$, then $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ exists and is equal to $A^{n-1}f/\|A^{n-1}f\|$ by the power

series expansion of $T(t)$ in I.3 and by Lemma 3.6 applied to $f(t) = \frac{1}{(n-1)!}(tA)^{n-1}$ and $g(t) = \sum_{m\le n-2}(tA)^m$.

Let now $0<f$ and $A^n f \neq 0$ for all $n\in\mathbb{N}$. To a given $\varepsilon>0$ we find $n'\in\mathbb{N}$ and $0<g\in E$ such that

$$\left\| \frac{A^n f}{\|A^n f\|} - g \right\| < \varepsilon \quad \text{for all } n>n'. \tag{3.3}$$

Let $t'>1$ such that

$$\left(\sum_{n\le n'} \|A\|^n\right)\cdot\|f\|\cdot\sup_{n\le n'}\left(\left\|\frac{A^n f}{\|A^n f\|} - g\right\|\right) < t'\cdot\varepsilon\cdot\frac{\|A^{n'+1}f\|}{(n'+1)!} \tag{3.4}$$

Since E is an AL-space and since $A\ge 0$ we obtain

$$\left\|\frac{T(t)f}{\|T(t)f\|} - g\right\| = \frac{1}{\|T(t)f\|}\cdot\left\|T(t)f - \left(\sum_{n=0}^{\infty}\left\|\frac{(tA)^n}{n!}f\right\|\right)g\right\| \le$$

$$\frac{1}{\|T(t)f\|}\sum_{n=0}^{\infty}\left[\left\|\frac{(n!)^{-1}(tA)^n f}{\|(n!)^{-1}(tA)^n f\|} - g\right\|\cdot\|(n!)^{-1}(tA)^n f\|\right].$$

In view of (3.3) and (3.4) this term is less or equal to

$$\frac{1}{\|T(t)f\|}\cdot\sum_{n>n'}\left(\varepsilon\left\|\frac{(tA)^n}{n!}f\right\|\right) + \frac{1}{\|T(t)f\|}\cdot\frac{t'\varepsilon\|A^{n'+1}f\|}{(n'+1)!}\cdot\frac{}{\sum_{n\le n'}\|A\|^n\|f\|}\cdot\sum_{n\le n'}\left\|\frac{(tA)^n f}{n!}\right\|$$

$$\le \varepsilon + \frac{(n'+1)!}{t^{n'+1}\|A^{n'+1}f\|}\cdot\frac{t'\cdot\varepsilon\|A^{n'+1}f\|}{(n'+1)!\cdot\sum_{n\le n'}\|A\|^n\|f\|}\cdot\sum_{n\le n'}\frac{t^n}{n!}\|A^n f\|$$

$$\le \varepsilon + \frac{t'\varepsilon}{t} \le 2\varepsilon \quad \text{for } t>\max(1,t').$$

□

The assumption of the preceding theorem is e.g. fulfilled in the situation of Theorem 3.3 on AL-spaces.

III.3 Applications

Example 3.11

Consider the semigroup T of the second section of chapter II. We claim that $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ exists for all $f>0$ and is equal to e_1. By Lemma 3.6 and formula (2.5) we only have to show that for all $f>0$:

$$\langle f,e_1\rangle \neq 0$$

and

$$\frac{\|\sum_{n=2}^{\infty} e^{-n^2\pi^2 t}\langle f,e_n\rangle e_n\|}{\|e^{-\pi^2 t}\langle f,e_1\rangle e_1\|} \to 0 \text{ as } t\to\infty.$$

The first assertion is obvious since $e_1(x) > 0$ for $0<x<1$, and therefore the second assertion follows immediately.

By the same argument we see that T is irreducible: Assume that T is not irreducible, then $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|} = e_1$ is contained in a closed ideal different from E, but in $L^2[0,1]$ all order ideals have the form $\{f\in E : f(x)=0 \text{ for all } x\in S\}$, where $S \subset [0,1]$ has positive Lebesgue measure [20, III.1].

In fact, all the assumptions of Corollary 3.9 are fulfilled:

T is differentiable since obviously $T(t)f\in D(A)$ for all $f\in E$, $t>0$;

and $AT(t)f = \sum_{n=1}^{\infty} (-n^2\pi^2)e^{-n^2\pi^2 t}\langle f,e_n\rangle e_n$, hence T is eventually norm continuous. The resolvent of A is compact and given by (c.f. [6, 2.40])

$$R(\lambda,A)f = \sum_{n=1}^{\infty} (\lambda+n^2\pi^2)^{-1}\cdot\langle f,e_n\rangle e_n ,$$

and therefore $s(A) = -\pi^2$ is a pole of the resolvent of A.

Example 3.12

We consider the preceding example with Neumann instead of Dirichlet boundary conditions, i.e.
$E = L^2[0,1]$, $D(A) = \{f\in E : f''\in E,\ f'(0)=f'(1)=0\}$, $Af = f''$. Then by [6, 2.40] (A,D(A)) generates a semigroup $T = (T(t))_{t\geq 0}$ which is given by

$$T(t)f = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}\langle f,e_n\rangle e_n \quad \text{for } t\geq 0 , \tag{3.5}$$

where

$$e_n(x) = \begin{cases} \sqrt{2}\cdot\cos(n\pi x) & \text{for } n>0 \\ 1 & \text{for } n=0, \end{cases} \quad 0\leq x\leq 1,$$

and $\langle f,e_n\rangle = \int_0^1 f(x)e_n(x)dx$.
The semigroup is positive by the Jacobi identity (see [16, A-I,2.7]). Similarily as in the preceding example we can show that $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|} = e_0$ exists for all $f>0$, that T is irreducible, differentiable and has compact resolvent.
By [16, A-I,2.7] (see also [15, pp. 175-177]) the same is true if we consider (A,D(A)) on $C[0,1]$ instead of $L^2[0,1]$, the only difference is that (3.5) is only valid for $t>0$.

Example 3.13: In [10] the following linear transport equation is discussed

$$\frac{\partial}{\partial t}\phi(x,v,t) = -\sum_{i=1}^{3} v_i \cdot \frac{\partial}{\partial x_i}\phi(x,v,t) - \sigma(x,v)\cdot\phi(x,v,t) + \int_V \chi(x,v,v')\cdot\phi(x,v',t)dv' , \tag{3.6}$$

$$\phi(x,v,0) = \phi_o(x,v),$$

for $x \in X \subset \mathbb{R}^3$, $v \in V \subset \mathbb{R}^3$, where X is a compact subset of $\mathbb{R}^3$ with non-empty interior, $V := \{v \in \mathbb{R}^3 : \|v\| := (\sum_{i=1}^{3} v_i^2)^{1/2} \leq v_{max}\}$, and $\sigma : X\times V \to \mathbb{R}$, $\chi : X\times V\times V \to \mathbb{R}$ are positive bounded measurable functions.

On $E := L^1(X\times V)$ (3.6) yields the following abstract Cauchy problem:

$$\frac{d}{dt}f(t) = Af(t) ,$$

$$f(0) = f_o,$$

where

$$A := A_o - M_\sigma + K_\chi,$$

$$D(A_o) := \{f \in E : -\sum_{i=1}^{3} v_i \cdot \frac{\partial}{\partial x_i} f \in E\}, \quad A_o f := -\sum_{i=1}^{3} v_i \cdot \frac{\partial}{\partial x_i} f,$$

$$M_\sigma f := \sigma \cdot f, \quad K_\chi f := \int_V \chi(\cdot,\cdot,v')f(\cdot,v')dv'.$$

Let $\lambda^* := \lim_{n\to\infty}(\text{ess inf}\{\sigma(x,v) : x\in X, \|v\| \leq \frac{1}{n}\})$, where ess inf denotes the essential infimum (see e.g. [25, I.3,Ex. 3]). Then it is shown in [10] that $s(A) \geq -\lambda^*$. Moreover $s(A) = \omega(A)$ is a

strictly dominant eigenvalue of A and pole of the resolvent of A if $s(A) > -\lambda^*$. The positive semigroup generated by (A,D(A)) is irreducible if the function χ fulfills certain conditions ([10, 2.5]) indepent of σ.

Assume that $s(A) > -\lambda^*$ and that T is irreducible. Then by [10, 2.3] $e^{-t\cdot s(A)}T(t)$ converges in norm as $t\to\infty$ to a strictly positive projection P of rank 1 (c.f. section 1 of this chapter). In particular the assumptions of Theorem 3.7 are fulfilled and $\lim_{t\to\infty}\frac{T(t)f}{\|T(t)f\|}$ exists for all $f>0$.

CHAPTER IV

CONTROLLABILITY

IV.1 Introduction

Controllability of linear systems of the form

$$\frac{d}{dt}f(t) = Af(t) + Bu(t),\ f(0) = 0,$$

with no restrictions on the control u has been investigated in the past both for finite dimensional and infinite dimensional state spaces. We will recall some of the results on exact and approximate controllability in the second section and we will add a new result (Prop. 4.6) in the spirit of this thesis' subject.

Saperstone and Yorke [18] and Brammer [4] characterized controllability with restrained controls on finite dimensional spaces. Korobov and Son [13], [22] and the author [21] could generalize these results to infinite dimensional spaces for exact controllability. In the third section we review these results and show that a corresponding characterization for approximate controllability is not true.

IV.2 Controllability without Restrictions on the Controls

Let E and U be Banach spaces, $(A,D(A))$ the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on E and $B\in\mathcal{L}(U,E)$. We consider the following linear system

$$\begin{aligned} \frac{d}{dt}f(t) &= Af(t) + Bu(t) \ , \\ f(0) &= f_o \ , \end{aligned} \tag{4.1}$$

where $f(t)\in E$, $f_o\in E$, $u(t)\in U$.

By [6, 2.23] for every $u\in L^1_{loc}(\mathbb{R}_+)$ a mild solution of (4.1) is given by

$$f(t) = T(t)f_o + \int_0^t T(t-s)Bu(s)ds \ . \tag{4.2}$$

In view of this one defines the following.

Definition 4.1

Let $1\leq p\leq\infty$ and consider (4.1).

We call E the state space and U the input space.

Define for $t>0$ the set of reachable states from the origin ($f_o=0$) in time t as

$$R_t := \{\int_0^{t'} T(t'-s)Bu(s)ds : u\in L^p([0,t'];U),\ 0\leq t'\leq t\} \ ,$$

and the set of reachable states from the origin in arbitrary time as

$$R = \cup_{t>0} R_t .$$

The pair (A,B) is called

exactly controllable in time t if $R_t = E$,

exactly controllable if $R = E$,

approximately controllable in time t if R_t is dense in E,

approximately controllable if R is dense in E.

In view of Baire's theorem (A,B) is exactly controllable for some $t>0$ if it is exactly controllable. The notion of approximate controllability is invariant under change of p and we can even restrict the controls to C^∞-functions. In finite dimensional spaces the situation is rather simple, and we have the following characterization.

Proposition 4.2 ([5, 11.4])
If dim $E = n < \infty$, then all controllability notions in 4.1 are equivalent to the surjectivity of the operator

$$B \oplus AB \oplus A^2B \oplus \ldots \oplus A^{n-1}B : U^n \to E . \qquad (4.3)$$

If dim $U = m < \infty$ this means that the $n \times nm$ - matrix $(B,AB,A^2B, \ldots ,A^{n-1}B)$ has rank n.

In infinite dimensional spaces we have a similar characterization for bounded A.

Proposition 4.3 ([12], [23])

Let E, U be Banach spaces, $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(U,E)$, then

(a) (A,B) is exactly controllable if and only if there exists $n \in \mathbb{N}$ such that the operator defined in (4.3) is surjective.

(b) (A,B) is approximately controllable (in time t) if and only if the linear span of $(BU, ABU, A^2BU, \ldots)$ is dense in E.

In general the following characterization is quite useful. Here we denote by E', U', B' and T(t)' the dual spaces, respectively adjoint maps of E, U, B and T(t), and by B'T(·)'f' the function $\mathbb{R}_+ \to U'$, $t \to B'T(t)'f'$.

Proposition 4.4 ([6, 3.7, 3.11 and proof of 3.15])

Consider the control system (4.1).

(a) If E and U are reflexive Banach spaces and $1<p<\infty$ in 4.1 then (A,B) is exactly controllable in time t if and only if there exists $k>0$ such that $\|B'T(\cdot)'f'\|_{L^q([0,t];U')} > k\|f'\|$ for every $f' \in E'$, where $p^{-1}+q^{-1}=1$.

(b) (A,B) is approximately controllable in time t if and only if the following implication holds:
B'T(s)'f' = 0 for all $0 \leq s \leq t$ implies f'=0.

(c) (A,B) is approximately controllable if and only if the following implication holds:
B'T(s)'f' = 0 for all $0 \leq s$ implies f'=0.

As a consequence of 4.4 (b) and (c) one can obtain characteri-

zations of approximate controllability for unbounded operators A, similar to those for bounded operators in 4.3 (b), see e.g. [24].

Finally we mention results which show that the concept of controllability is too strong for many infinite dimensional systems.

Proposition 4.5 ([6, 3.2], [23], [17, pp. 101-105])
Consider the control problem (4.1). (A,B) is not exactly controllable if E is infinite dimensional and if one of the following two conditions holds.

(i) B is the uniform limit of operators with finite rank (especially, if E has the approximation property and if B is compact).

(ii) E is reflexive and B is compact.

Exact controllability is also not possible in the following case.

Proposition 4.6
Let E and U be Banach lattices, $T = (T(t))_{t\geq 0}$ the generator of a positive semigroup on E with generator (A,D(A)) and $B \in \mathcal{L}(U,E)$ positive. Then (A,B) is not exactly controllable if U is isomorphic (as a Banach lattice) to some C(X), X compact (in particular if U is finite dimensional) and if E is not isomorphic to any C(X), X compact.

Proof: A real Banach lattice is isomorphic to some C(X) if and only if its positive cone has interior points [20, II.7]. Assume that (A,B) is exactly controllable. We show that the set

$$R_t^+ := \{\int_0^t T(t-s)Bu(s)ds : u\in L^p([0,t];U),\ u(s)\in U_+,\ 0\le s\le t\}$$

has non-empty interior in E (cf. [13, proof of Theorem 5]). Since by 4.1 $E = R = \bigcup_{t>0} R_t$ and since $R_s \subset R_t$ for $s\le t$, Baire's theorem implies that R_t is of second category for some $t>0$. Hence the map

$$\Phi : L^p([0,t];U) \to E,\ u \to \int_0^t T(t-s)Bu(s)ds$$

which has image R_t is an open map. Now the set

$$L^p([0,t];U_+) := \{u\in L^p([0,t];U) : u(s)\in U_+,\ 0\le s\le t\}$$

has non-empty interior in (the real part of) $L^p([0,t];U)$ since $\operatorname{int} U_+ \neq \emptyset$. Hence $R_t^+ = \Phi(L^p([0,t];U_+))$ has non-empty interior in (the real part of) E. Since $R_t^+ \subset E_+$ this leads to a contradiction.

□

IV.3 Controllability with Restrictions on the Controls

We now will discuss how the concept of controllability changes if we impose restrictions on the controls. Consider again the control system (4.1) and assume that the control $u(\cdot)$ takes values in a subset Ω of U. We define the following.

Definition 4.7

Let $1 \leq p \leq \infty$ and consider (4.1). For $t>0$ the set of reachable states from the origin in time t by means of controls which take values in Ω is defined as

$$R_{\Omega,t} = \{\int_0^{t'} T(t'-s)Bu(s)ds : u \in L^p([0,t'];U),\ u(s) \in \Omega,\ 0 \leq s \leq t' \leq t\},$$

and the set of reachable states from the origin in arbitrary time by means of controls which take values in Ω as

$$R_\Omega = \cup_{t>0} R_{\Omega,t}.$$

The pair (A,B) is called

exactly Ω-controllable in time t if $R_{\Omega,t} = E$,

exactly Ω-controllable if $R_\Omega = E$,

approximately Ω-controllable in time t if $R_{\Omega,t}$ is dense in E,

approximately Ω-controllable if R_Ω is dense in E.

Moreover (A,B) is called

locally exactly Ω-controllable in time t if $R_{\Omega,t}$ is a neighborhood of zero in E,

locally exactly Ω-controllable if R_Ω is a neighborhood of zero,

locally approximately Ω-controllable in time t if $R_{\Omega,t}$ is dense in a neighborhood of zero,

locally approximately Ω-controllable if R_Ω is dense in a neighborhood of zero.

Again (local) exact Ω-controllability implies by Baire's theorem (local) exact Ω-controllability in time t for some $t>0$. Each of the Ω-controllability notions is implied by the corresponding local Ω-controllability if Ω is a cone. In finite dimensional spaces $R_{\Omega,t}$ is always convex [4], therefore the corresponding exact and approximate Ω-controllability notions coincide. Moreover the following theorem holds.

Theorem 4.8 ([4])

Let (4.1) be given on a finite dimensional space E, let $\Omega \cap \operatorname{Ker} B \neq \emptyset$. Then

(A,B) is locally (exactly or approximately) Ω-controllable (in time $t>0$) if and only if

(A,B) is controllable in any sense of Definition 4.1 and there is no eigenvector f' to a real eigenvalue of A' satisfying $\operatorname{Re}\langle f', Bu\rangle > 0$ for all $u \in \Omega$.

For infinite dimensional systems there is the following generalization of 4.8.

Theorem 4.9 ([13], [21], [22])

Consider the control system (4.1) and let $\Omega \cap \operatorname{Ker} B \neq \emptyset$.

(a) Let $\operatorname{int}\Omega \neq \emptyset$. Then (A,B) is locally exactly Ω-controllable if and only if

(A,B) is exactly controllable and there is no eigenvector f' as in 4.8.

(b) Let $\mathrm{int}(\mathrm{co}\Omega) \neq \emptyset$. Then (A,B) is locally approximately Ω-controllable if (A,B) is exactly controllable and if there is no eigenvector f' as in 4.8.

The main lack in 4.9 is that (A,B) is assumed to be exactly controllable which in view of 4.5 (a) is a strong requirement.
In fact, the following example shows that a corresponding characterization to 4.9 (a) for local approximate controllability does not hold.

Example 4.10

Let $E = L^p([0,\frac{1}{2}])$, $1 \leq p < \infty$, $T = (T(t))_{t \geq 0}$ the semigroup on E given by

$$T(t)f(x) = \begin{cases} f(x+t) & \text{for } x+t \leq 1/2 \\ 0 & \text{for } x+t > 1/2 \end{cases};$$

let $U = l^\infty$, $\Omega = U_+$ its canonical positive cone, and for $u = (u_0, u_1, \dots) \in l^\infty$ let

$$(Bu)(x) = \sum_{i=0}^{\infty} u_i x^i \text{ for } 0 \leq x \leq \tfrac{1}{2}.$$

Since $\sum_{i=0}^{n} |u_i| x^i \leq \|u\| \cdot \sum_{i=0}^{n} x^i \leq \|u\|$, the limit $\sum_{i=0}^{\infty} u_i x^i$ exists, and $Bu \in E$ with $\|Bu\| = (\int_0^{1/2} (\sum_{i=0}^{\infty} u_i x^i)^p dx)^{1/p} \leq (\int_0^{1/2} \|u\|^p dx)^{1/p} = (\frac{1}{2}\|u\|^p)^{1/p} \leq \|u\|$. Hence $B \in \mathcal{L}(U,E)$. Moreover BU contains all polynomials, therefore BU is dense in E and (A,B) is approximately controllable in every time $t>0$ by Proposition 4.4 (b).
(A,B) is not approximately Ω-controllable (and therefore not locally approximately Ω-controllable by the remark to Definition

4.7): BU_+ is contained in the positive cone E_+ of E, $T(t)E_+ \subset E_+$ for all $t \geq 0$ and therefore $cl(R_\Omega) \subset cl(E_+) = E_+ \neq E$.

However, the following assertions hold:

$\Omega \cap \mathrm{Ker}\ B \neq \emptyset$, $(1,1, \ldots) \in \mathrm{int}\Omega \neq \emptyset$, and there exists no eigenvector f' as in 4.8, since for the adjoint of the generator $(A, D(A))$ of T we have

$D(A') = \{f \in L^q([0, \frac{1}{2}]) : f \text{ is absolutely continuous, } f(0)=0\}$,

where $q^{-1} := 1 - p^{-1}$,

$A'f(x) = -\frac{d}{dx}f(x)$ and $p\sigma(A') = \emptyset$.

CHAPTER V

POSITIVE CONTROLLABILITY

V.1 Introduction

In this chapter we treat the first question stated in the second chapter. In the preceding chapter IV characterizations for controllability of linear systems were given, both for the case that controls were allowed to take all values in the control space or that they were restricted to a particular subset of it. Controllability was defined in the sense that we wanted to reach at least a dense subset of the entire state space, respectively of a neighborhood of zero. However, in many instances for systems with restrictions on the controls it is a priori known that all reachable states are contained in a closed cone C of the state space (as in the examples of the second chapter). In this case controllability in the sense of chapter IV is impossible but it is interesting to know conditions under which the reachable states are dense in C (which was just our question in chapter 2). In the second section we will characterize systems for which the reachable states are contained in the positive cone of a Banach lattice, and then we try to find for a particular class conditions under which a dense subset of the positive cone can be reached. We define this as positive controllability and we will

see that this implies the usual controllability as defined in chapter IV. Our first characterization for positive controllability will be a result analogous to Proposition 4.4. However, this characterization is not so useful as the corresponding one in chapter IV. But by the general results obtained in the second section and by some technical lemmas in the appendix we will be able to give feasible criterions for positive controllability for finite dimensional state spaces in section 3. In section 4 we present strong necessary conditions for infinite dimensional state spaces and finite dimensional input spaces. We conclude this chapter with some applications.

V.2 General Results

Let E be a Banach lattice, U a Banach space, $(A,D(A))$ the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on E and $B\in\mathcal{L}(U,E)$. As in (4.1) we consider the linear control system

$$\begin{aligned} \frac{d}{dt}f(t) &= Af(t) + Bu(t) \ , \\ f(0) &= f_o \ , \end{aligned} \tag{5.1}$$

where $u\in L^1_{loc}(\mathbb{R}_+,U)$ as in IV.3 takes values in a subset Ω of U and f_o is in the positive cone E_+ of E.
First we want to know conditions under which the solution f(t) remains in E_+ for all such u and f_o. We state the following proposition.

Proposition 5.1

Let E be a Banach lattice, U a Banach space, (A,D(A)) the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on E, $B\in\mathcal{L}(U,E)$ and $0\in\Omega\subset U$. The following assertions are equivalent.

(i) The mild solutions of (5.1) remain in E_+ for every $f_o\in E_+$ and every $u\in L^1_{loc}(\mathbb{R}_+,U)$ with $u(t)\in\Omega$ for $t\geq 0$.

(ii) T is a positive semigroup and $B\Omega\subset E_+$.

Proof: The implication (ii)→(i) follows immediately from the variation of constants formula (4.2).

(i)→(ii): Since $0\in\Omega$ we can take $u(\cdot)$ identical zero, then $f(t) = T(t)f_o \in E_+$ for all $f_o\in E_+$, i.e. T is positive. On the other hand taking $f_o = 0$, $u(s) = \frac{u}{t}$ for some $u\in\Omega$ and all $0\leq s\leq t$, we obtain by (4.2) $f(t) = \frac{1}{t}\int_0^t T(s)Bu\,ds$ which tends to Bu as $t\to 0$. Closedness of E_+ implies $Bu\in E_+$.

After this characterization we restrict ourselves to the case where U is also a Banach lattice, Ω the positive cone U_+ of it and therefore B a positive operator by the preceding proposition. We then define positive controllability as follows.

Definition 5.2

Let E and U be Banach lattices, (A,D(A)) the generator of a positive semigroup $T = (T(t))_{t\geq 0}$ on E and $B\in\mathcal{L}(U,E)$ positive. For $t>0$ and $1\leq p\leq\infty$ the set of reachable states from the origin in time t by means of non-negative controls u is defined as

$$R_t^+ := \{\int_0^{t'} T(t'-s)Bu(s)ds : u\in L^p([0,t];U),\ u(s)\in U_+,\ 0\leq s\leq t'\leq t\},$$

and the set of reachable states from the origin in arbitrary time by means of non-negative controls u as

$$R^+ := \cup_{t>0} R_t^+.$$

The pair (A,B) is called

<u>exactly positive controllable in time t</u> if $R_t^+ = E_+$,

<u>exactly positive controllable</u> if $R^+ = E_+$,

<u>approximately positive controllable in time t</u> if $cl(R_t^+) = E_+$,

<u>approximately positive controllable</u> if $cl(R^+) = E_+$.

Of course, each of the notion implies the corresponding controllability notion in Definition 4.1 since $R_t = R_t^+ - R_t^+$. Moreover we remark that R_t^+ is just the set $R_{\Omega,t}$ of Definition 4.7 for $\Omega = U_+$. The notion of approximate positive contollability (in time t) is invariant under change of p and we can even restrict the controls to C^∞-functions.

We give some simple examples.

<u>Example 5.3</u>

(a) Let E = U be a Banach lattice, A = 0, B = Id. Then obviously (A,B) is exactly positive controllable in any time t>0.

(b) Let $E = U = L^p(\mathbb{R}_+)$, $1 \leq p < \infty$ and $\mathcal{T}$ the translation semigroup on E given by $(T(t)f)(x) = f(x+t)$ for all $x,t \geq 0$. Let $(A,D(A))$ be the generator of it and $B = Id$. Then (A,B) is exactly positive controllable in any time $t>0$.

Proof: A strongly continuous semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ on E is given by $S(t)f(x) = \begin{cases} f(x-t) & \text{for } x \geq t \\ 0 & \text{for } x<t \end{cases}$, and $T(t)S(t) = Id$ for all $t \geq 0$. For $f \in E_+$ we set $u(s) := \frac{1}{t}S(t-s)f$ and obtain $f = \int_0^t \frac{1}{t} f \, ds = \int_0^t T(t-s)u(s)ds \in R_t^+$.

Example 5.4

Let $E = \mathbb{R}^2$, $U = \mathbb{R}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (A,B) is exactly controllable in any time $t>0$ by Proposition 4.2. However, (A,B) is not approximately positive controllable: Since $\exp(tA) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(2t) \end{pmatrix}$ we see that $R^+ \subset \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}_+^2 : a \leq b\}$.

Our first attempt to give a characterization of approximate positive controllability is the following result analogous to 4.4.

Proposition 5.5

Let the assumptions be as in Definition 5.2.

(a) (A,B) is approximately positive controllable in time t if and only if the following implication holds:

$B'T(s)'f' \geq 0$ for all $0 \leq s \leq t$ implies $f' \geq 0$. (5.2)

(b) (A,B) is approximately positive controllable if and only if the following implication holds:

$B'T(s)'f' \geq 0$ for all $0 \leq s$ implies $f' \geq 0$.

Proof: (a) Let (A,B) be approximately positive controllable in time $t>0$ and $B'T(s)'f' \geq 0$ for $0 \leq s \leq t$. We have to show $f' \geq 0$. For every $u \in L^p([0,t];U)$ taking values in U_+ we have $\langle f', T(t-s)Bu(s)\rangle = \langle B'T(t-s)'f', u(s)\rangle \geq 0$ for $0 \leq s \leq t$ and therefore $\langle f', \int_0^t T(t-s)Bu(s)ds\rangle \geq 0$, hence $\langle f', f\rangle \geq 0$ for all $f \in cl(R_t^+) = E_+$, i.e. $f' \geq 0$.

On the other hand let the implication (5.2) hold. Suppose that there exists $f_o \in E_+ \setminus cl(R_t^+)$. By the second separation theorem ([19, II.9.2]) there exists $\alpha \in \mathbb{R}$, $f' \in E'$ (resp. $f' \in E_{\mathbb{R}}'$ if E is a complex Banach lattice) such that

$$\langle f', f_o\rangle < \alpha < \langle f', f\rangle \text{ for all } f \in cl(R_t^+).$$

Since $cl(R_t^+)$ is a cone containing zero we see that $\alpha < 0$ and

$$\langle f', f\rangle \geq 0 \text{ for all } f \in cl(R_t^+). \tag{5.3}$$

In particular $\langle f', f_o\rangle < 0$, hence $f' \neq 0$ and this implies by (5.2) that $B'T(s_o)'f' \neq 0$ for some $0 \leq s_o \leq t$. Hence $\langle B'T(s_o)'f', u_o\rangle < 0$ for some $u_o \in U_+$ and therefore $\langle B'T(s_o)'f', u_o\rangle < 0$ for all s in a ε-neighborhood $K_\varepsilon(s_o)$ of s_o. Let $u \in L^p([0,t];U)$ be given by

$$u(t-s) = \begin{cases} -\langle B'T(s)'f', u_o\rangle \cdot u_o & \text{for } s \in K := K_\varepsilon(s_o) \cap [0,t] \\ 0 & \text{otherwise} \end{cases}.$$

Then u takes values in U_+, hence $\int_0^t T(t-s)Bu(s)ds \in cl(R_t^+)$ and

$$\langle f', \int_0^t T(t-s)Bu(s)ds\rangle = \int_0^t \langle B'T(s)'f', u(t-s)\rangle ds =$$

$$-\int_K \langle B'T(s)'f', u_o\rangle^2 ds < 0$$

in contradiction to (5.3). Therefore $cl(R_t^+) = E_+$.
The proof of (b) is analogous to (a).

□

Proposition 4.4 is the basis for other characterizations of approximate controllability in chapter IV, e.g. 4.2 and 4.3 is due to the fact that for a bounded generator A the assertion $B'T(s)'f' = 0$ for all $0 \leq s$ is equivalent to $B'A'^n f' = 0$ for all $n \in \mathbb{N}$ (by the power series expansion of $T(t)'$). However, Proposition 5.5 does not have such nice consequences: The assertion $B'T(s)'f' \geq 0$ for all $0 \leq s$ does by no means imply that $B'A'^n f' \geq 0$ for all $n \in \mathbb{N}$.

In the following we will no longer make use of 5.5 but follow another approach to obtain better characterizations of positive controllability. We begin with the following proposition which gives a useful representation of the sets $cl(R_t^+)$ and $cl(R^+)$. For a subset M of a Banach space E let coM, coconeM be defined as in chapter I.

Proposition 5.6

Let the assumptions be as in Definition 5.2.

(a) $cl(R_t^+) = cl(co\{T(s)Bu : 0 \leq s \leq t,\ u \in U_+\})$,

$cl(R^+) = cl(co\{T(s)Bu : 0 \leq s,\ u \in U_+\})$.

(b) Let $U = \mathbb{R}^m$ and let $e(1), \ldots, e(m)$ be the canonical unit vectors of $\mathbb{R}^m$. Then

$cl(R_t^+) = cl(cocone\{T(s)Be(i) : 0 \leq s \leq t,\ 1 \leq i \leq m\})$,

$cl(R^+) = cl(cocone\{T(s)Be(i) : 0 \leq s,\ 1 \leq i \leq m\})$.

Proof: (a) By definition of the integral we see that the sets on the left hand side are contained in the sets on the right. On the other hand R_t^+, R^+ are convex since U_+ is convex; it remains to show that $T(t)Bu \in cl(R_t^+)$ for all $t>0$, $u \in U_+$. For this purpose let $u \in U_+$ and $u_n(s) = \begin{cases} nu & \text{for } 0 \le s \le 1/n \\ 0 & \text{for } 1/n \le s \le t \end{cases}$. Then $u_n(\cdot) \in L^1([0,t];U)$ and $\| \int_0^t T(t-s)Bu(s)ds - T(t)Bu \| \le n \cdot \int_0^{1/n} \| T(t-s)Bu - T(t)Bu \| ds \to 0$ as $n \to \infty$.

(b) Since $B\mathbb{R}_+^m = \text{cocone}\{Be(1),\ldots,Be(m)\}$ we see that (a) implies (b).

□

We apply Proposition 5.6 to the following example.

Example 5.7

Let $E = U = l^p$, $1 \le p < \infty$, and let the operators A, B be given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & & & & \cdots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & & & & & & \vdots \end{pmatrix}.$$

Obviously $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(U,E)$ and A generates the positive semigroup $T = (T(t))_{t \ge 0}$ given by

$$T(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & t & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & t & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & & \vdots & & & & \end{pmatrix}.$$

Let $e(1), e(2), \ldots$ denote the canonical unit vectors of E and U. Then for $i \in \mathbb{N}$: $T(0)Be(i) = e(2i)$, $\lim_{t\to\infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|} = e(2i-1)$, hence $cl(R^+) = E_+$ by Proposition 5.6 (a), i.e. (A,B) is approximately positive controllable. However, for all $t>0$ the ratio of the 2i-th coordinate of $\int_0^t T(t-s)Bu(s)ds$ to the (2i-1)-th coordinate is greater than $\frac{1}{t}$, hence $e(2i-1)$ cannot be contained in $cl(R_t^+)$ or R^+, i.e. (A,B) is neither approximately positive controllable in any time $t>0$ nor exactly positive controllable.

V.3 Finite Dimensional Case

In this section we will discuss positive controllability in the case where both the state space and the input space are finite dimensional, and we will obtain a characterization for approximate positive controllability.

Theorem 5.8

Let the assumptions be as in Definition 5.2, $E = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $e[1], \ldots, e[n], e(1), \ldots, e(m)$ the canonical unit vectors of E resp. U.

(a) (A,B) is approximately positive controllable in time $t>0$ if and only if

for all $1 \le i \le n$ there exists $1 \le j \le m$ and $\mu > 0$ such that

$$e[i] = \mu Be(j). \tag{5.4}$$

(b) (A,B) is approximately positive controllable

if and only if

for all $1 \le i \le n$ there exists $1 \le j \le m$ and $\mu > 0$ such that

$$e[i] = \mu Be(j) \text{ or } e[i] = \lim_{t\to\infty} \frac{T(t)Be(j)}{\|T(t)Be(j)\|} . \tag{5.5}$$

Proof: The sufficiency of (a) and (b) is obvious by Proposition 5.6 (b) and Theorem 3.5.

For the necessity we may assume as in the proof of Lemma A.2 in the appendix that $Be(j) \neq 0$ for all $1 \le j \le m$.

(a) If (A,B) is approximately positive controllable in some time $t > 0$ then by Lemma A.2 (a)

$$\{T(s)Be(j) : 0 \le s \le t, 1 \le j \le m\} \cap J_{f'} \neq \emptyset \quad \text{for all} \quad J_{f'} \neq \{0\} \tag{5.6}$$

Since $A + \|A\| Id \ge 0$ ([16, C-II,1.11]) we obtain for every $t > 0$ and $1 \le j \le m$

$$\begin{aligned} e^{t\|A\|} \cdot T(s)Be(j) &= Be(j) + \sum_{n=1}^{\infty} \frac{1}{n!} (t(A+\|A\|Id))^n Be(j) \\ &\ge Be(j) . \end{aligned} \tag{5.7}$$

Now (5.6) and (5.7) yield $\{Be(j): 1 \le j \le m\} \cap J_{f'} \neq \emptyset$ for all $J_{f'} \neq \{0\}$. This implies the assertion (5.4).

(b) If (A,B) is approximately controllable then by Lemma A.2 (b) and Theorem 3.5

$$[\{T(s)Be(j) : 0 \le s, 1 \le j \le m\} \cup \{\lim_{t\to\infty} \frac{T(t)Be(j)}{\|T(t)Be(j)\|} : 1 \le j \le m\}] \cap J_{f'} \neq \emptyset.$$

for all $J_{f'} \neq \{0\}$. Again by (5.7) we obtain

$$[\{Be(j) : 1 \le j \le m\} \cup \{\lim_{t\to\infty} \frac{T(t)Be(j)}{\|T(t)Be(j)\|} : 1 \le j \le m\}] \cap J_{f'} \neq \emptyset$$

for $J_{f'} \neq \{0\}$, and this implies the assertion (5.5).

□

Corollary 5.9

Let the assumptions be as in the preceding theorem.

(a) (A,B) is approximately positive controllable in time $t>0$ if and only if $BU_+ = E_+$. In particular, approximate positive controllability in time $t>0$ implies $m \geq n$.

(b) If (A,B) is approximately positive controllable then $2m \geq n$, and for every $1 \leq i \leq n$ the following holds: $e[i]$ is an eigenvector of A or $e[i] = \mu Be(j)$ for some $1 \leq j \leq m$, $\mu > 0$.

(c) If (A,B) is exactly positive controllable then (A,B) is approximately positive controllable in any time $t>0$.

Proof: (a) If $Bu = e[i]$ for $u>0$ then $\mu Be(j) = e[i]$ for some $\mu>0$, $1 \leq j \leq m$ since B is positive. The assertion follows from 5.8 (a).

(b) This is immediate from 5.8 and 3.4.

(c) We show that (5.4) holds. Since (A,B) is exactly positive controllable, there exists for all $J_f, \neq \{0\}$ and $0<f \in J_f$, a time $t>0$ such that $f \in R_t^+$, and by Lemma A.2 (a) we obtain that $\{T(s)Be(j):0 \leq s \leq t, 1 \leq j \leq m\} \cap J_f, \neq \emptyset$. As in the proof of 5.8 (a) this yields (5.4).

□

We point out that 5.8 and 5.9 show that the different positive controllability notions do not coincide on finite dimensional state spaces contrary to the usual controllability notions (Proposition 4.2).

As an example for these results we mention again 5.4. By 5.8 (b)

it is immediate that (A,B) in 5.4 is not approximately positive controllable. More examples will be discussed in the fifth section.

V.4 Infinite Dimensional Case

In this section we will discuss positive controllability for the case where the state space is infinite dimensional and the input space is finite dimensional, as is the case in most practical applications.

The results of Corollary 5.9 for finite dimensional state spaces may lead to the conjecture that positive controllability in the above situation is impossible. In fact, we will show that approximate positive contollability in finite time $t>0$ is never possible for a large class of systems on nearly all state spaces of practical interest. Moreover, on the same state spaces approximate positive controllability is impossible for systems for which (as in the finite dimensional case) $\lim_{t\to\infty} \frac{T(t)Bf}{\|T(t)Bf\|}$ exists for all $f>0$. However, as we pointed out in chapter III, the limit may not exist, and in this case there exists an example where the system is approximately positive controllable.

Finally we should mention that by Proposition 4.5 and by the remark to Definition 5.2 exact positive controllability is never

possible in the case where the state space is infinite dimensional and the input space finite dimensional.

We state the following theorem.

Theorem 5.10

Let E be an infinite dimensional Banach lattice and let there exist a strictly positive linear form on E. Let $(A,D(A))$ be the generator of a positive semigroup $T = (T(t))_{t>0}$ on E, let $B \in L(\mathbb{R}^m, E)$ be positive, $m \in \mathbb{N}$, let $e(1), \ldots, e(m)$ denote the canonical unit vectors of $\mathbb{R}^m$ and assume $Be(j) \neq 0$ for $1 \leq j \leq m$.

(a) (A,B) is not approximately positive controllable in time $t>0$ if $T(s)Be(j) \neq 0$ for every $0 \leq s \leq t$ and $1 \leq j \leq m$.

(b) (A,B) is not approximately positive controllable if $T(s)Be(j) \neq 0$ for every $0 \leq s$ and $1 \leq j \leq m$ and if $\lim_{t\to\infty} \frac{T(t)Be(j)}{\|T(t)Be(j)\|}$ exists for every $1 \leq j \leq m$.

Before proving the theorem we remark the following.

Remark 5.11

(a) The assumption that there exists a strictly positive linear form is fulfilled for all state spaces of practical interest, e.g. $C_o(X)$, X a locally compact subset of $\mathbb{R}^n$, $L^p(\Omega,\lambda)$, Ω a measurable subset of $\mathbb{R}^n$, λ the Lebesgue measure.

(b) The assumption $Be(j)\neq 0$ in the theorem is just for convenience. If $Be(j)=0$ for some j, then the conditions in (a) and (b) of the theorem read

$0 \notin \{T(s)Be(j) : 0\leq s\leq t,\ 1\leq j\leq m,\ Be(j)\neq 0\}$ resp.

$0 \notin \{T(s)Be(j) : 0\leq s,\ 1\leq j\leq m,\ Be(j)\neq 0\}$, $\lim_{t\to\infty} \frac{T(t)Be(j)}{\|T(t)Be(j)\|}$ exists for all $1\leq j\leq m$ for which $Be(j)\neq 0$.

For the proof of this fact we restrict B to the sublattice of $\mathbb{R}^m$ generated by those $e(j)$ for which $Be(j)\neq 0$, and apply then 5.10.

(c) The condition in 5.10 (a) as well as the first condition in 5.10 (b) is obviously fulfilled by every positive semigroup which can be extended to a (not necessarily positive) strongly continuous group, and also by every holomorphic positive semigroup: Let $T = (T(t))_{t\geq 0}$ be a positive holomorphic semigroup and $T(t)f=0$ for some $t>0$, $f\in E$, then $T(s)f=0$ for $s\geq t$, hence $\langle f', T(s)f\rangle = 0$ for every $s\geq t$ and $f'\in E'$. Since T is holomorphic this implies that $\langle f', T(s)f\rangle = 0$ for all $s>0$ and $f'\in E'$, hence $\langle f', f\rangle = 0$ and $f=0$.

Proof of 5.10: (a) As mentioned in the proof of Lemma A.2 (a) the set $C := \{T(s)Be(j) : 0\leq s\leq t,\ 1\leq j\leq m\}$ is compact and does not contain 0 by assumption. On the other hand by Lemma A.5 there exist infinitely many pairwise orthogonal ideals $J_{f'} \neq \{0\}$, $0<f'\in E'$, and by Lemma A.3 C can only intersect finitely many of them. By Lemma A.2 (a) this implies that $cl(R_t^+)$ has non-trivial intersection with only finitely many of these ideals, hence $cl(R_t^+) \neq E_+$, i.e. (A,B) is not approximately positive controllable in time t.

(b) Again as mentioned in the proof of Lemma A.2 (b) the assumption implies that

$$\{\frac{T(s)Be(j)}{\|T(s)Be(j)\|} : 0\leq s,\ 1\leq j\leq m\}\cup\{\lim_{t\to\infty}\frac{T(t)Be(j)}{\|T(t)Be(j)\|} : 1\leq j\leq m\}$$

is compact and does not contain 0. Lemma A.5, A.3 and A.2 (b) yield in the same way as in (a) that $cl(R^+) \neq E_+$, i.e. (A,B) is not approximately positive controllable.

□

We do not know whether 5.10 holds on all Banach lattices. We also lack a counterexample to the conjecture that (A,B) is never approximately positive controllable in any time $t>0$ if E is infinite dimensional and $U = \mathbb{R}^m$. But the following example shows that (A,B) may be approximately positive controllable if the limit in the assumption of 5.10 (b) does not exist. The basic idea of the example is due to J. Zabczyk.

Example 5.12

(a) Let $E = L^p(\mathbb{R}_+)$, $1\leq p<\infty$, $U = \mathbb{R}$, $T = (T(t))_{t\geq 0}$ the (positive) translation semigroup on E given by

$$(T(t)f)(x+t) = f(x+t) \quad \text{for} \quad x,t\geq 0$$

with generator (A,D(A)), and let $B\in L(\mathbb{R},E)$ be a positive operator given by

$$Bu = u\cdot b \text{ for some } b\in E_+.$$

We claim that (A,B) is approximately positive controllable for a suitable $b \in E_+$, which we will construct in the following.

First we know that there exists a countable subset S of normalized positive functions with compact support such that $cl(coconeS) = E_+$: Take e.g. all characteristic functions on intervals $[p,p+q]$, $p,q \in \mathbb{Q}_+ \setminus \{0\}$, divided by their L^p-norm. Then we can also find a sequence $f_1, f_2, \ldots$ in S such that each element of S occurs infinitely many times in this sequence.

By assumption the support $spt(f_i)$ is contained in $[0,t_i]$ for some $t_i > 0$. Define

$$s(1) := 0, \quad s(n+1) := \sum_{i=1}^{n} t_i \quad \text{for} \quad n \geq 1$$

and let

$$f_i^{(s)} \text{ denote the function } t \to \begin{cases} f_i(t-s) & \text{for } t \geq s \\ 0 & \text{for } t < s \end{cases} .$$

Since $spt(f_i^{(s(i))}) \cap spt(f_j^{(s(j))}) \subset [s(i),s(i+1)] \cap [s(j),s(j+1)]$ has measure zero for $i \neq j$, we can define

$$b := \sum_{i=1}^{\infty} 2^{-i^2} \cdot f_i^{(s(i))}$$

and obtain $\|b\| \leq (\sum_{i=1}^{\infty} 2^{-pi^2})^{1/p} \leq 1$, hence $b \in E_+$.

Because of Proposition 5.6 (a) (A,B) is approximately positive controllable for this b if $S \subset cl\{T(s)bu : 0 \leq s, u \geq 0\}$.

Therefore we show now that for each $f \in S$ and $n \in \mathbb{N}$ there exists $s,u > 0$ such that

$$\|T(s)bu - f\| < 2^{-n} . \tag{5.8}$$

Since f occurs infinitely many times in the sequence (f_i) we can find $k>n$ such that $f=f_k$. Let $s=s(k)$, $u=2^{k^2}$. Then by definition of T we obtain

$$T(s)f^{(s(i))} = T(s(i)+t_i+\ldots+t_{k-1})f^{(s(i))} = 0 \text{ for } i<k,$$
$$T(s)f^{(s(k))} = f,$$
$$T(s)f^{(s(i))} = T(s(k))f^{(s(k)+s(i)-s(k))} = f^{(s(i)-s(k))} \text{ for } i>k.$$

Hence

$$T(s)bu - f = \sum_{i=k+1}^{\infty} 2^{k^2-i^2}.f^{(s(i)-s(k))}.$$

Since

$$\operatorname{spt}(f_i^{(s(i)-s(k))}) \cap \operatorname{spt}(f_j^{(s(j)-s(k))}) \subset$$
$$[s(i)-s(k), s(i+1)-s(k)] \cap [s(j)-s(k), s(j+1)-s(k)]$$

has measure zero for $i \neq j$, we obtain

$$\|T(s)bu - f\| \leq \left(\sum_{i=k+1}^{\infty} 2^{p(k^2-i^2)}\right)^{1/p} =$$
$$= \left(\sum_{i=k+1}^{\infty} 2^{p(k+i)(k-i)}\right)^{1/p} = \left(\sum_{i=1}^{\infty} 2^{-p(2k+i)i}\right)^{1/p}$$
$$\leq \left(\sum_{i=1}^{\infty} 2^{-p2k}.2^{-pi^2}\right)^{1/p} \leq 2^{-2k}\left(\sum_{i=1}^{\infty} 2^{-pi^2}\right)^{1/p}$$
$$\leq 2^{-2k} < 2^{-n}.$$

Thus (5.8) is proven.

□

(b) Nearly the same example as in (a) can be constructed on $E = C_o(\mathbb{R}_+)$. Let T be as in (a). As countable subset S of E_+ we

take e.g. all those polygones of supnorm 1 on $\mathbb{R}_+$ which have a finite number of "edges" with coordinates $(0,q_o)$, (p_1,q_1), ... , (p_{n-1},q_{n-1}), $(p_n,0)$, where $p_i,q_i \in \mathbb{Q}$, $n \in \mathbb{N}$. It is not difficult to show that cl(coconeS) = E_+. Define the sequence (f_i) as in (a) with $spt(f_i) \subset [0,t_i]$ and let $s(1) := 0$, $s(n+1) = \sum_{i=1}^{n} (t_i+1)$. Then $spt(f_i^{(s(i))}) \cap spt(f_j^{(s(j))})$ is void for $i \neq j$ and we can define b as in (a) with $\|b\| = \frac{1}{2}$. If we define s, u as in (a) we obtain again $\|T(s)bu - f\| \leq u \cdot 2^{-(k+1)^2} = 2^{-2k+1} < 2^{-n}$.

V.5 Applications

In this section we apply the main theorems 5.8 and 5.10 of the preceding two sections to some concrete examples.

Example 5.13

The electrically heated oven of II.1 is neither approximately positive controllable in any time $t>0$ nor exactly positive controllable because of 5.9 (a), (c).

It is approximately positive controllable by 5.8 (b) if and only if

$$\binom{0}{1} = \lim_{t\to\infty} \frac{T(t)\binom{1}{0}}{\|T(t)\binom{1}{0}\|} . \tag{5.9}$$

By 3.4 a necessary condition for (5.9) is that $\binom{0}{1}$ is an eigenvector to A, i.e. $\beta = 0$. One verifies that then

$$T(t) = \begin{pmatrix} e^{-\alpha t} & 0 \\ \gamma t e^{-\alpha t} & e^{-\delta t} \end{pmatrix} \quad \text{if } \alpha = \delta, \qquad \text{and}$$

$$T(t) = \begin{pmatrix} e^{-\alpha t} & 0 \\ \gamma(e^{-\alpha t} - e^{-\delta t}) \cdot (\delta - \alpha)^{-1} & e^{-\delta t} \end{pmatrix} \quad \text{if } \alpha \neq \delta.$$

Hence (5.9) holds if and only if

$$\gamma \neq 0, \ \beta = 0, \ \delta \leq \alpha.$$

In the physical situation (c.f. II.1) we have $\alpha, \beta, \gamma, \delta > 0$, and therefore the oven is not approximately positive controllable. But by 4.2 (A,B) is (exactly) controllable if and only if the matrix $\begin{pmatrix} 1 & -\alpha \\ 0 & \gamma \end{pmatrix}$ is non-singular, i.e. $\gamma \neq 0$.

Example 5.14

In [8] the following system as an approximation for a distributed network is discussed.

$$\frac{d}{dt} f(t) = A_o f(t) + \mathrm{Id}[\psi(f_1(t)), \ \ldots \ , \psi(f_n(t))] \ , \qquad (5.10)$$

where

$$A_o = \begin{pmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ 0 & 0 & 1 & -2 & \ldots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \ldots & & -2 & 1 \\ 0 & 0 & \ldots & & 1 & -2 \end{pmatrix} ,$$

$f(t) = (f_1(t), \ \ldots \ , f_n(t)) \in \mathbb{R}^n =: E$, and ψ a real function on $\mathbb{R}$. If ψ is linear then (5.10) reads as

$$\frac{d}{dt}f(t) = A_o f(t) + \mu Idf(t) \qquad (5.11)$$

for some $\mu \in \mathbb{R}$.

Define $A := A_o + \mu Id$, then A by [16, B-II,1.4 (b)] generates a positive semigroup, and we can consider the linear control system (5.1) for some positive $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $m \in \mathbb{N}$.

Since no canonical unit vector e[i] of $\mathbb{R}^n$ is an eigenvector of A we infer from Proposition 3.4 that the second condition in Theorem 5.8 (b) can never be fulfilled. Hence (A,B) is approximately positive controllable if and only if it is approximately positive controllable for any time $t > 0$, i.e. if and only if $BU_+ = E_+$ (by 5.9 (a)).

However, from 4.2 it is immediate that (A,B) is controllable (in the usual sense) if BU contains e.g. e[1] or e[n], in particular B may even have rank 1.

Example 5.15

Another approximation for a certain distributed network in [8] is the following

$$\frac{d}{dt}f(t) = A_o f(t) + [\psi(f_n(t)), 0, \ldots, 0] , \qquad (5.12)$$

where

$$A_o = \begin{pmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ 0 & 0 & 1 & -2 & \ldots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \ldots & & -2 & 1 \\ 0 & 0 & \ldots & & 1 & -1 \end{pmatrix} ,$$

$f(t) = (f_1(t), \ldots, f_n(t)) \in \mathbb{R}^n := E$, and ψ is a real function on

$\mathbb{R}$. If ψ is linear then (5.12) reads as

$$\frac{d}{dt}f(t) = A_\mu f(t) \tag{5.13}$$

for some $\mu \in \mathbb{R}$, where

$$A_\mu = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & \mu \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 \\ & & \ddots & & & \\ 0 & 0 & \dots & & -2 & 1 \\ 0 & 0 & \dots & & 1 & -1 \end{pmatrix},$$

A_μ generates by [16, B-II,1.4 (b)] a positive semigroup if $\mu \geq 0$, and we can consider the linear control system (5.1) for some positive $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $m \in \mathbb{N}$.

Since no canonical unit vector e[i] of $\mathbb{R}^n$ is an eigenvector of A, we infer again that (A,B) is approximately positive controllable if and only if it is approximately positive controllable for any time $t > 0$. In particular this is not the case for $B : \mathbb{R} \to \mathbb{R}^n$, $\lambda \to \lambda(1,0, \dots ,0)$ (unless n=1) which will be the natural control operator in this example. However, by 4.2 (A,B) is controllable for this B.

Example 5.16

We consider the heat equation on a finite rod with Dirichlet boundary conditions as in II.2.

As we have seen in Example 3.11 $\lim_{t\to\infty} \frac{T(t)f}{\|T(t)f\|}$ exists for every $f > 0$, in particular $T(t)f \neq 0$ for all $t \geq 0$, $f > 0$. Since there exists a strictly positive linear form on $L^2[0,1]$ we obtain by 5.10 that

the control system in II.2 is not approximately positive controllable.

In the same way we can show using the results of Example 3.12 that the system also is not approximately positive controllable if we replace the Dirichlet by Neumann boundary conditions.

Example 5.17

Consider the Laplacian on $\mathbb{R}^n$. This means we consider the control problem (5.1) for $E = C_o(\mathbb{R}^n)$ or $E = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $U = \mathbb{R}^m$, $0 < B \in L(U,E)$, $D(A) = \{f \in E : \Delta f \in E\}$, $Af = \Delta f$. By [16, C-II, 1.5 (c), (d) and remark] $(A,D(A))$ generates a positive contraction semigroup $T = (T(t))_{t \geq 0}$ on E which is given by

$$T(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t) \cdot f(y)\,dy,$$

where $|x-y|$ denotes the Euclidean norm of $x-y$ in $\mathbb{R}^n$.

This shows that $T(t)f(x) > 0$ for all $x \in \mathbb{R}^n$ whenever $f > 0$, in particular T is irreducible [20, III.1], and the assumptions of Theorem 5.10 (a) are fulfilled. Hence (A,B) is not approximately positive controllable in any time $t > 0$.

On the other hand (A,B) may be approximately controllable in every time $t > 0$ even for $U = \mathbb{R}^2$ (see [6, 3.17]).

Example 5.18

Example 5.14 was an approximation for the following system

$$\frac{\partial}{\partial t}\chi(t,x) = \frac{\partial^2}{\partial x^2}\chi(t,x) + f(\chi(t,\cdot))(x), \quad 0 \leq x \leq 1,$$
$$\chi(t,0) = \chi(t,1) = 0,$$

where f is a map on the space of functions on [0,1].

If f is scalar multiple of the identity, then the semigroup associated with this system is just the one given in (2.5) multiplied by scalars $e^{-\mu t}$. For finite dimensional positive control we therefore obtain the same assertions as in Example 5.16 for the heat equation on a finite rod with Dirichlet boundary conditions.

If f is an arbitrary positive linear map and continuous on $L^2[0,1]$, then the associated system on $L^2[0,1]$ yields a positive semigroup $S = (S(t))_{t\geq 0}$, which by the perturbation formula in [16, A-II,1.29] dominates T given in (2.5), i.e. $S(t)-T(t)$ is positive for all $t\geq 0$. In particular $S(t)f > 0$ for every $f>0$ (see 5.16), and by Theorem 5.10 (a) positive controllability in time $t>0$ is impossible for finite dimensional positive control.

CHAPTER VI

POSITIVE STATIONARY PAIRS

VI.1 Introduction

In the forgoing chapter we obtained rather negative answers to the first question posed in chapter II. However, it is often not so important to reach the entire positive cone of the state space. It suffices to design a non-negative control by which particular positive states can be approximated and held constant for all times. This was just the problem formulated in the second question of chapter II for positive states belonging to positive stationary pairs. Moreover we observed in chapter II a relationship between existence of a positive stationary pair and stability of the system. In the present chapter we will discuss these problems systematically.

Let E and U be Banach lattices, $T = (T(t))_{t>0}$ a positive semigroup on E with generator (A,D(A)) and take $B \in \mathcal{L}(U,E)$ positive. As in the previous chapters we consider the control problem

$$\begin{aligned} \frac{d}{dt}f(t) &= Af(t) + Bu(t) \ , \\ f(0) &= f_o, \end{aligned} \tag{6.1}$$

where $u \in L^1_{loc}(\mathbb{R}_+,U)$ takes values in U_+ and $f_o \in E_+$.

A mild solution is given by $f(t) = T(t)f_o + \int_0^t T(t-s)Bu(s)ds$.

We call a pair $(f_1,u_1) \in (E_+\setminus\{0\}) \times U_+$ positive stationary if $Af_1 + Bu_1 = 0$. In this case $f(\cdot) \equiv f_1$ is a non-zero constant solution of (6.1) for $u(\cdot) \equiv u_1$, $f_o = f_1$. We are interested whether each $f_1 \in E_+$ belonging to a positive stationary pair can be (approximately) reached and held constant by a non-negative control $u(\cdot)$. Moreover, we study the connection between existence of positive stationary pairs and stability of the semigroup T as defined in chapter I.

Our main results will be formulated in Theorem 6.1 and Theorem 6.6. Further results and applications will be given in the third and fourth section.

VI.2 Main Results

We state the following theorem.

Theorem 6.1

Let E and U be Banach lattices, $T = (T(t))_{t\geq 0}$ a uniformly exponentially stable positive semigroup on E with generator (A,D(A)) and take $B \in \mathcal{L}(U,E)$ positive. Then

(a) To each $u_1 \in U_+\setminus \mathrm{Ker}B$ there exists exactly one $f_1 \in E_+$ such that (f_1,u_1) is a positive stationary pair.

(b) If (f_1,u_1) is a positive stationary pair, $f_o \in E_+$ and $u(\cdot) \equiv u_1$, then the solution of (6.1) tends to f_1 as $t \to \infty$.

Proof: (a) If T is uniformly exponentially stable then $0 \in \rho(A)$ and $-A^{-1}$ is positive by [16, C-III,1.1]. For all $u_1 \in U_+ \setminus \mathrm{Ker}B$ we obtain therefore that $(-A^{-1}Bu_1, u_1)$ is a positive stationary pair. On the other hand $0 \in \rho(A)$ implies that to each $u_1 \in U_+$ there exists at most one f_1 such that (f_1, u_1) is a positive stationary pair.

(b) For a positive stationary pair (f_1, u_1) and $u(\cdot) \equiv u_1$ we obtain for the mild solution of (6.1): $f(t) = T(t)f_o - \int_0^t T(t-s)Af_1 ds$. This is equal to $T(t)(f_o - f_1) + f_1$, and therefore tends to f_1 as $t \to \infty$ since T is stable.

□

From the above proof we obtain the following.

Corollary 6.2

Let E and U be Banach lattices, $T = (T(t))_{t \geq 0}$ a weakly stable positive semigroup on E with generator $(A, D(A))$ and take $B \in \mathcal{L}(U,E)$ positive. Then

(a) To each $u_1 \in U_+ \setminus \mathrm{Ker}B$ there exists at most one $f_1 \in E_+$ such that (f_1, u_1) is a positive stationary pair; such $f_1 \in E_+$ exists if $0 \in \rho(A)$.

(b) If (f_1, u_1) is a positive stationary pair, $f_o \in E_+$ and $u(\cdot) \equiv u_1$, then the solution of (6.1) tends in the weak topology of E to f_1 as $t \to \infty$; the convergence holds in the norm-topology if T is strongly stable.

Proof: (a) Uniqueness of f_1 follows from the fact that A is injective if T is weakly stable. Moreover weak stability implies that the growth bound $\omega(A)$ of A is not positive and therefore by [16, C-III,1.1] the resolvent $R(\lambda,A)$ of A in λ is a positive operator for all $\lambda>0$. Thus if $0\in\rho(A)$ we have $-A^{-1} = \lim_{\lambda\to 0} R(\lambda,A) \geq 0$, and as in the proof of 6.1 (a) we obtain the existence of a positive stationary pair.

(b) This follows immediately from the proof of 6.1 (b).

□

For another immediate consequence of 6.1 we remind the definition of R^+ from 5.2.

Corollary 6.3

Let E and U be Banach lattices, $T = (T(t))_{t\geq 0}$ a positive semigroup on E with generator $(A,D(A))$ and take $B\in\mathcal{L}(U,E)$ positive. Then $\{R(\lambda,A)Bu: u\in U_+, \lambda>\omega(A)\} \subset cl(R^+)$.

Proof: Obviously R^+ does not change if we multiply $T(t)$ by $e^{-\lambda t}$ for $\lambda>\omega(A)$. Then $(e^{-\lambda t}T(t))_{t\geq 0}$ is an exponentially stable semigroup with generator $A_\lambda := A-\lambda$, and $-(A_\lambda)^{-1} = R(\lambda,A)$. The assertion then follows from the proof of 6.1.

□

The following example shows that there may exist no positive stationary pair if the semigroup T is only strongly stable.

Example 6.4

We consider the Laplacian on $E = C_o(\mathbb{R})$ or $L^p(\mathbb{R})$, $1 \le p < \infty$ as in Example 5.17 with $B \in \mathcal{L}(\mathbb{R}^m, E)$ positive. The semigroup T is strongly stable ([16, A-IV,1.2]). We claim that there exists no positive stationary pair.

Suppose that $Af+Bu = 0$ for some $0 \le f \in D(A)$, $u \ge 0$. We show $f=0$.

The equation $Af+Bu = 0$ implies that the second derivative f'' of f takes only non-positive values. Now for some constants c_1, $c_2 \in \mathbb{R}$ the function f is given by

$$f(x) = c_2 + c_1 x + \int_0^x \int_0^y f''(z)\,dz\,dy \ , \ x \in \mathbb{R} \ . \tag{6.2}$$

Let g be the function which maps y on $\int_0^y f''(z)\,dz$. Then $g(0)=0$ and g is antitone. Since $g(y) \le 0$ for $y \ge 0$ and since $f \ge 0$ we obtain by (6.2) (if we choose x large) that $c_1 \ge 0$. On the other hand $g(y) \ge 0$ for $y \le 0$ implies that $\int_0^x \int_0^y f''(z)\,dz\,dy = -\int_x^0 g(y)\,dy \le 0$ for $x \le 0$. (6.2) and $f \ge 0$ imply $c_1 \le 0$, hence $c_1 = 0$.

Since g is antitone $\alpha := \lim_{y \to -\infty} g(y)$ and $\beta := \lim_{y \to +\infty} g(y)$ exist in the extended real field and $\alpha \ge 0$, $\beta \le 0$. Assume $\alpha > 0$ then $\int_0^x \int_0^y f''(z)\,dz\,dy = -\int_x^0 g(y)\,dy \to -\infty$ as $x \to -\infty$ which contradicts the fact that $f \ge 0$ and $c_1 = 0$. In the same way $\beta < 0$ is impossible. Thus $\alpha = \beta = 0$, i.e. $g=0$, hence $f(x) = c_2$ for all x. Since $f \in C_o(\mathbb{R})$ or $f \in L^p(\mathbb{R})$ this is only possible if $f=0$.

□

Thus we have shown that strong stability is not sufficient to ensure the existence of a positive stationary pair. On the other

hand the following example shows that not even boundedness of the semigroup is necessary for the existence of such a pair.

Example 6.5

Let $E = \mathbb{R}^2$, $U = \mathbb{R}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1)$ is a positive stationary pair, but $(e^{tA})_{t\geq 0}$ is an unbounded semigroup since $1 \in p\sigma(A)$.

The following theorem states under what conditions the existence of a positive stationary pair implies boundedness or even stability of the semigroup T.

Theorem 6.6

Let E and U be Banach lattices, $T = (T(t))_{t\geq 0}$ a positive semigroup on E with generator $(A, D(A))$. Take $B \in \mathcal{L}(U,E)$ positive and let (f_1, u_1) be a positive stationary pair.

(a) T is bounded if $f_1 \in \text{int}E_+$.

(b) T is stable if E is finite dimensional, $f_1 \in \text{int}E_+$ and $Bu_1 \in \text{int}E_+$.

Proof: (a) If f_1 is an interior point of E_+, then to each $f \in E_+$ there exists $\lambda_f > 0$ such that $f \leq \lambda_f \cdot f_1$. Since $f_1 - T(t)f_1 = -\int_0^t T(s)Af_1 ds = \int_0^t T(s)Bu_1 ds \geq 0$, we obtain $0 \leq T(t)f \leq \lambda_f T(t)f_1 \leq \lambda_f f_1$ and $\|T(t)f\| \leq \lambda_f \|f_1\|$ for all $t \geq 0$. By the principle of uniform boundedness T is bounded.

(b) Since $\|T(t)f\| \leq \lambda_f \|T(t)f_1\|$ for every $f \in E_+$, it suffices to show that $T(t)f_1 \to 0$ as $t \to \infty$.

Now $T(t)f_1 = f_1 - \int_0^t T(s)Bu_1 ds$, hence

$$T(r+t)f_1 = T(r)f_1 - \int_r^{t+r} T(s)Bu_1 ds \leq T(r)f_1 .$$

Thus $(T(t)f_1)_{t\geq 0}$ is decreasing and therefore converges as $t\to\infty$ to some $g\in E_+$ since E is finite dimensional ([20, II.5.11]). Then g is a fixed vector of every $T(t)$, and $0 \leq g \leq T(t)f_1$ for all $t\geq 0$. Moreover $Bu_1 \in \text{int}E_+$ implies $\frac{1}{t}\int_0^t T(s)Bu_1 ds \in \text{int}E_+$ for small enough $t>0$. Since

$$\frac{1}{t}(f_1-g) \geq \frac{1}{t}(f_1-T(t)f_1) = \frac{1}{t}\int_0^t T(s)Bu_1 ds$$

we obtain $\frac{1}{t}(f_1-g)\in \text{int}E_+$. Thus $f_1-g\in \text{int}E_+$, hence there exists $\mu>0$ such that $f_1 \leq \mu(f_1-g)$ and therefore

$$0 \leq T(t)f_1 \leq \mu T(t)(f_1-g) = \mu(T(t)f_1-g) \to 0 \text{ as } t\to\infty.$$

This implies that $T(t)f_1 \to 0$ as $t\to\infty$.

□

We remark that Theorem 6.6 can be applied only to spaces E which are isomorphic to C(X), X compact, since for all other space $\text{int}E_+=\emptyset$ (see proof of 4.6).

As we already saw in Example 6.5 T may not be bounded if (f_1,u_1) is a positive stationary pair with $f_1 \notin \text{int}E_+$. Moreover T may not be strongly stable if $f_1 \in \text{int}E_+$ and $Bu_1 \notin \text{int}E_+$ as the following example shows.

Example 6.7

Let $E = \mathbb{R}^2$, $U = \mathbb{R}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, 1)$ is a positive stationary pair, but $(\exp(tA))_{t\geq 0}$ is not stable since $0 \in p\sigma(A)$.

VI.3 Further Results

In the previous section we showed some basic relations between existence of a positive stationary pair (f_1, u_1), reachability of f_1 and stability of the underlying semigroup T. However, some interesting questions are still open and will be considered in the following.

Let us assume that a positive stationary pair (f_1, u_1) exists. In 6.1 and 6.2 we saw that f_1 can be approximately reached by means of the constant control u_1 if T is strongly stable. We ask the following:

1) Under what conditions can we reach f_1 exactly by means of a non-negative control u?
2) If T is not strongly stable or even not bounded, can we still approximately reach f_1 by means of a non-negative control u?

Concerning the first question we state the following proposition.

Proposition 6.8

Let E and U be Banach lattices, E finite dimensional, $\mathcal{T}=(T(t))_{t\geq 0}$ a stable positive semigroup on E with generator (A,D(A)). Take $B\in\mathcal{L}(U,E)$ positive and let (f_1,u_1) be a positive stationary pair. If (A,B) is controllable (as defined in 4.1, 4.2) and $u_1\in \text{int}U_+$ then to each neighborhood $\Omega\subset U_+$ of u_1 there exists $t_o>0$ and a control $u\in L^\infty_{loc}(\mathbb{R}_+,U)$ taking values in Ω such that the mild solution f(t) of (6.1) is equal to f_1 for $t\geq t_o$.

Proof: Let $\varepsilon>0$ be given such that an ε-neighborhood of u_1 is contained in U_+. By [5, proof of 11.3] we know that for controllable (A,B) and given $t_1>0$ there exists $\delta>0$ with the property that for all $f\in E$ with $\|f\|<\delta$ there exists $u\in L^\infty([0,t_1];U)$ with $\|u(s)\|<\varepsilon$ for $0\leq s\leq t_1$ and

$$\int_o^{t_1} T(t_1-s)Bu(s)ds = f \ . \tag{6.3}$$

In other words, we know that there is a continuous linear operator which maps each $f\in E$ on a suitable control $u\in L^\infty([0,t_1];U)$ such that u steers the origin to f in time t_1.
Since $\mathcal{T}$ is stable there exists $t_2>0$ such that

$$\|T(t)(f_1-f_o)\| < \delta \text{ for } t>t_2.$$

By (6.3) we have $v\in L^\infty([0,t_1];U)$ with $\|v(s)\|<\varepsilon$ for $0\leq s\leq t_1$ and

$$\int_o^{t_1} T(t_1-s)Bv(s)ds = T(t_1+t_2)(f_1-f_o).$$

Choose

$$u(s) = \begin{cases} u_1 & \text{for } 0 \le s \le t_2 \text{ and } s > t_1+t_2 \\ u_1+v(s-t_2) & \text{for } t_2 < s \le t_1+t_2 \end{cases}.$$

Then $u \in L^{\infty}_{loc}(\mathbb{R}_+, U_+)$ takes values in the ε-neighborhood of u_1, and for $t \ge t_1+t_2 =: t_o$ we obtain

$$T(t)f_o + \int_0^t T(t-s)Bu(s)ds =$$

$$T(t)f_o - \int_0^t T(t-s)Af_1 ds + \int_{t_2}^{t_1+t_2} T(t-s)Bv(s-t_2)ds =$$

$$T(t)f_o - (T(t)f_1-f_1) + \int_0^{t_1} T(t-t_2-s)Bv(s)ds =$$

$$f_1 - T(t)(f_1-f_o) + T(t-t_2-t_1)\int_0^{t_1} T(t_1-s)Bv(s)ds =$$

$$f_1 - T(t)(f_1-f_o) + T(t-t_2-t_1)T(t_1+t_2)(f_1-f_o) = f_1.$$

□

To our second question we obtain a negative answer.

Proposition 6.9

Let E and U be Banach lattices, $\mathcal{T}=(T(t))_{t\ge 0}$ a positive semigroup on E with generator $(A,D(A))$. Take $B \in \mathcal{L}(U,E)$ positive and let (f_1,u_1) be a positive stationary pair.

(a) If $\mathcal{T}$ is an unbounded semigroup, then in every neighborhood of f_1 there exists $f_o \in E_+$ such that for every positive control the mild solution of (6.1) does not tend to f_1 as $t \to \infty$.

(b) If T is bounded but not strongly stable, then there exists $f_o \in E_+$ such that for every positive control the mild solution of (6.1) does not tend to f_1 as $t \to \infty$.

Proof: We begin with the following preliminary remark: If T is an unbounded positive semigroup then by the uniform boundedness theorem $\limsup_{t\to\infty} \|T(t)f\| = \infty$ for some $f \in E_+$.
If T is bounded but not strongly stable then $\limsup_{t\to\infty} \|T(t)f\| > 0$ for some $f \in E_+$.
(a) Let $\varepsilon > 0$ be given, $\limsup_{t\to\infty} \|T(t)f\| = \infty$ for some $f \in E_+$ and $f_o := f_1 + \frac{\varepsilon \cdot f}{2\|f\|}$. Then $f_o \in E_+$ is in the ε-neighborhood of f_1 and for $u \in L^1_{loc}(\mathbb{R}_+, U)$ with positive values we obtain for the mild solution of (6.1):

$$f(t) = T(t)f_o + \int_0^t T(t-s)Bu(s)ds \geq T(t)f_o \geq \frac{\varepsilon T(t)f}{2\|f\|} \geq 0,$$

hence

$$\limsup_{t\to\infty} \|f(t)\| \geq \frac{\varepsilon}{2\|f\|} \cdot \limsup_{t\to\infty} \|T(t)f\| = \infty \quad \text{and} \quad f(t) \not\to f_1.$$

(b) Let $s := \limsup_{t\to\infty} \|T(t)f\| > 0$ for some $f \in E_+$ and $f_o := \frac{2f}{s}\|f_1\| \in E_+$. Then for $u \in L^1_{loc}(\mathbb{R}_+, U)$ with positive values we obtain for the mild solution of (6.1):

$$f(t) = T(t)f_o + \int_0^t T(t-s)Bu(s)ds \geq T(t)f_o \geq 0,$$

hence

$$\limsup_{t\to\infty} \|f(t)\| \geq \frac{2s}{s}\|f_1\| = 2\|f_1\| \text{ for all } t > 0 \quad \text{and} \quad f(t) \not\to f_1.$$

□

In the proof of the preceding proposition we chose f_o rather large. Therefore it is not surprising that f(t) cannot tend to f_1; but it is immediate that when we start in Example 6.7 from $f_o=0$ the state $f_1 = \binom{1}{1}$ cannot be reached (since $R^+ \subset \{\binom{a}{0} : a \geq 0\}$). However, (A,B) in that example is not controllable (in the sense of 4.1), and it is an open question whether controllability of (A,B) always implies that f_1 can be reached from the origin, i.e. $f_1 \in cl(R^+)$.

VI.4 Applications

Example 6.10

We again consider the electrically heated oven of II.1. As mentioned there, A is stable if and only if there exists a positive stationary pair (f_1,u_1) with $u_1>0$.

Since $\alpha,\beta,\gamma,\delta \geq 0$ it is immediate that A is stable if and only if $\det A > 0$. Then all positive stationary pairs are scalar multiples of $(\frac{1}{\det A}\binom{\delta}{\gamma},1)$.

Moreover we saw in II.1 that for $\gamma+\delta \neq 0$ a positive stationary pair $(x_1,0)$ exists if and only if A is not stable but has eigenvalues with non-positive real parts, i.e. $\det A = 0$. Then all positive stationary pairs are scalar multiples of $(\binom{\delta}{\gamma},0)$.

In the physical situation we have $\det A > 0$ (c.f. 5.13) and Theorem 6.1 is applicable. Moreover, as we saw in 5.13, (A,B) is controllable unless $\gamma = 0$, and therefore by Proposition 6.8 f_1 belonging to a positive stationary pair can be reached exactly.

Example 6.11

We consider the example of II.2. As shown there, the semigroup is uniformly exponentially stable and a positive stationary pair exists (c.f. Theorem 6.1 (a)). All f_1 belonging to a positive stationary pair (f_1,u_1) can be approximately reached by the control $u(\cdot) \equiv u_1$ according to 6.1 (b).

Example 6.12

We consider Example 3.12 where T is not stable but bounded. Let B be as in the preceding example, i.e. $B \in \mathcal{L}(\mathbb{R},E)$, $u \to u\cdot\psi$, $\psi \in E = L^2[0,1]$ or $C[0,1]$. A pair $(f_1,u_1) \in E\times\mathbb{R}$, $f_1 > 0$, $u_1 \geq 0$ is positive stationary if and only if

$$Af_1 + Bu_1 = 0 \qquad \text{if and only if}$$

$$f_1'' = -u_1\psi,\ f_1'(0) = f_1'(1) = 0 \qquad \text{if and only if}$$

$$f_1'(x) = -u_1\int_0^x \psi(y)\,dy,\ \int_0^1 \psi(y)\,dy = 0.$$

Since $\psi \geq 0$ this is only possible if $\psi = 0$, and then f_1 is a constant function.

Thus a positive stationary pair exists only in the trivial case that $B = 0$. Then the mild solution of (6.1) tends to f_1 if and only if $\langle f_o,e_o\rangle = \langle f_1,e_o\rangle$, where e_o denotes the constant one function as in 3.12.

CHAPTER VII

FEEDBACK CONTROLS

VII.1 Introduction

In the third question of chapter II we asked for particular positive feedback controls. This chapter will give some answers and at the same time we will discuss a more general problem. Its precise formulation will be presented in the next section. The problem itself is not restricted to positive systems but we will obtain much better results for those systems. In the third section we state a perturbation theorem which we will need for most of the results presented in the fourth and fifth section. We conclude the chapter with some remarks and applications.

Let us now describe our problem heuristically. We consider a linear system which tends to zero in the absence of controls and we assume that we can realize controls only in so called "feedback form". But controls of such a system may lead to unbounded solutions ("catastrophy"). Therefore it is desirable to design controls such that the system has the following properties:

There are initial states for which the system does not tend to zero, or even, the system has a constant solution different from zero,

and

the system remains bounded for all times, or even, for all initial states the system tends to a final state.

In the best case we expect that the controlled system tends to a non-zero final state for all "realistic" initial states.

Some answers to this problem were given in the previous chapter for abitrary controls but not for feedback controls.

VII.2 Formulation of the Problems

Let E, U be Banach spaces, (A,D(A)) the generator of a strongly continuous semigroup $T = (T(t))_{t \geq 0}$ on E and $B \in \mathcal{L}(U,E)$.
We consider the linear system

$$\begin{aligned} \frac{d}{dt} f(t) &= Af(t) + Bu(t) , \\ f(0) &= f_o , \end{aligned} \tag{7.1}$$

where $f_o \in E$, $u \in L^1_{loc}(\mathbb{R}_+, U)$.
Assume that the semigroup is stable as defined in chapter I (weakly, strongly or uniformly exponentially stable).
We ask whether there is a feedback control $u(t) = Cf(t)$, $C \in \mathcal{L}(E,U)$ such that the solution of (7.1) has the properties required in the first section. The solution of (7.1) for such a feedback

control is given by $f(t) = S(t)f_o$, where $\mathcal{S} := (S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by $(A+BC, D(A))$ on E. The questions read then as follows: Is there a $C \in \mathcal{L}(E,U)$ with the following properties:

$\mathcal{S}$ is not weakly stable , (7.2)

e. g. ,

$0 \in p\sigma(A+BC)$, (7.3)

and

there exists $M>0$ such that $\|S(t)\| \leq M$ for all $t\geq 0$. (7.4)

(7.3) and (7.4) are obviously fulfilled if

there exists a continuous projection $P \neq 0$ on E such that $S(t) \to P$ (in the weak, strong or norm topology) as $t\to\infty$. (7.5)

The range of such a projection P is the fixed space of $\mathcal{S}$ (i.e. $\cap_{t\geq 0}\mathrm{Ker}(\mathrm{Id}-S(t))$), resp. the kernel of A+BC.

Moreover, in case that (7.5) holds we wish that

P has rank 1 , (7.6)

and

$$Pf \neq 0 \text{ for all "realistic" } f \neq 0 \quad \text{(e.g. , for all } f \text{ in a cone } \Omega) \ . \tag{7.7}$$

If E is a Banach lattice and T a positive semigroup we can additionally ask whether C can be chosen such that

$$BC > 0 \text{ and therefore } S(t) \geq 0 \ . \tag{7.8}$$

If moreover U is a Banach lattice and $B>0$ we also wish that

$$C > 0 \ . \tag{7.9}$$

If E is a Banach lattice then (7.7) (for $\Omega=E_+$) means that P is strictly positive.

Furthermore in this case (7.6), (7.7) and (7.9) mean that there exists a positive linear feedback operator such that for all $f_o>0$ the solution of (7.1) tends to a positive scalar multiple of a state $0<f_1\in PE$. Then f_1 belongs to the positive stationary pair (f_1,Cf_1) (as defined in chapter VI), since $PE \subset \text{Ker}(A+BC)$ as mentioned above. Thus by (7.6), (7.7) and (7.9) we can find such a feedback operator for at least one f_1 belonging to a positive stationary pair. The question remains whether we can find it for all f_1 belonging to positive stationary pairs. If the resolvent $R(0,A)$ of A in 0 exists and is positive (e.g. if $s(A)<0$) then this question is equivalent to the fact that we can find a feedback operator to all elements in

$$R(0,A)(BU_+\setminus\{0\}), \tag{7.10}$$

since $Af_1+Bu_1 = 0$ if and only if $f_1 = R(0,A)Bu_1$.

Before we answer these questions we want to point out another interesting problem in this context.

Stability and boundedness of the semigroup S are closely related to the value of the growth bound $\omega(A+BC)$ of the semigroup (see [16, A.I]).

(7.2) implies that $\omega(A+BC) \geq 0$, (7.4) implies $\omega(A+BC) \leq 0$. This leads us to the following question: Let $\omega(A)<0$. For which C is

$$\omega(A+BC) \geq 0 \text{ or even } \omega(A+BC) = 0 \quad ? \tag{7.11}$$

VII.3 A Perturbation Theorem

We recall the definition of the essential spectral radius and the essential growth bound in chapter I, and we denote by $\mathcal{C}(E)$ the space of the compact linear continuous operators on a Banach space E endowed with the operator norm.

Proposition 7.1

Let $(A,D(A))$ be the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a Banach space E and let $K \in \mathcal{C}(E)$.

Then $(A+K,D(A))$ generates a strongly continuous semigroup $S_K = (S_K(t))_{t\geq 0}$ on E and the following assertions hold:

(a) For all $\varepsilon,t > 0$ the sets $\sigma(S_K(t))\cap\{\lambda : |\lambda|>e^{t(\omega(A)+\varepsilon)}\}$ and $\sigma(A+K)\cap\{\lambda : Re\lambda>\omega(A)+\varepsilon\}$ consist of a finite number of poles of the resolvents of $S_K(t)$, resp. $A+K$ with residues of finite rank.

(b) The following equalities hold

$\omega_{ess}(A) = \omega_{ess}(A+K)$, $\omega(A+K) = \max(\omega_{ess}(A+K),s(A+K))$.

If $\omega(A+K) > \omega(A)$ then $s(A+K) = \omega(A+K)$. Furthermore $\omega_{ess}(A) = \omega_{ess}(A_{|(Id-P)E})$ if P is a projection of finite rank which commutes with T.

(c) The map $\Phi : \mathcal{C}(E) \to \mathbb{R}\cup\{-\infty\}$, $K \to \omega(A+K)$ is continuous.

Proof: (a) By [9, 1.10] it is sufficient to show the assertion for $\sigma(S_K(t))$. From [16, A-III,3.7] we know that

$r_{ess}(S_K(t)) = \exp(t\cdot\omega_{ess}(A+K)) = \exp(t\cdot\omega_{ess}(A)) \leq \exp(t\omega(A))$

and $\sigma(S_K(t))\cap\{\lambda: |\lambda|>r_{ess}(S_K(t))+\delta\}$ consists of a finite number of poles with residues of finite rank.

(b) The first two assertions can be found in [16, A-III,3.7], and the last two are immediate consequences of them.

(c) Let K_n be a sequence of compact operators which converge in $\mathcal{C}(E)$ to K. We may assume $K = 0$ (otherwise consider $A+K$ instead of A). We show the following two inequalities which imply the

affirmed continuity of Φ.

$$\omega(A) \geq \limsup_{n\to\infty} \omega(A+K_n) \tag{7.12}$$

$$\omega(A) \leq \liminf_{n\to\infty} \omega(A+K_n) \tag{7.13}$$

Let $\omega > \omega(A)$, hence $\|T(t)\| \leq M_\omega e^{-\omega t}$ for some $M_\omega > 0$. It is well known [11, IX.2.1] that $\omega + M_\omega \cdot \|K_n\| \geq \omega(A+K_n)$ which implies (7.12). If $\omega(A) = \omega_{ess}(A)$ then by (b) $\omega(A) = \omega_{ess}(A+K_n) \leq \omega(A+K_n)$ for all n which implies (7.13). If $\omega(A) > \omega_{ess}(A)$ then $\omega(A) = s(A)$ by (b). Since the proof of (a) shows that for every $\varepsilon > 0$ the set $\sigma(A) \cap \{\lambda : \mathrm{Re}\lambda > \omega_{ess}(A)+\varepsilon\}$ consists of a finite number of poles of the resolvent of A with residues of finite rank, there has to exist such a pole λ_0 with $\mathrm{Re}\lambda_0 = \omega(A)$. By [11, IV.3.5] to each neighborhood U of λ_0 there is n_0 such that for all $n \geq n_0$ U contains a pole of the resolvent of $A+K_n$. Hence $\liminf_{n\to\infty} \omega(A+K_n) \geq \mathrm{Re}\lambda_0 = \omega(A)$.

□

As a corollary we obtain

Corollary 7.2

Let E be a Banach lattice, $(A,D(A))$ the generator of a positive semigroup $T = (T(t))_{t\geq 0}$ on E with $\omega(A) < 0$ and take $K \in \mathcal{C}(E)$ positive such that $\omega(A+K) = 0$. Then

(a) 0 is a pole of the resolvent of $A+K$ with residue P of finite rank.

(b) If $0 \in \sigma(A)$ has pole order 1 and one dimensional eigenspace, then $\sigma(A+K) \cap i\mathbb{R} = \{0\}$ and $\omega(A+K_{|(Id-P)E}) < 0$.

Proof: (a) By Proposition 7.1 (b) $s(A+K) = \omega(A+K) = 0$ and by [16, C-III,1.1] the spectral bound of a positive semigroup is contained in the spectrum of the generator; hence 7.1 (a) yields the assertion.

(b) If 0 is a pole of order 1 with one dimensional eigenspace then $\sigma(A+K)\cap i\mathbb{R}$ is an additive group by [9, 2.5 (b)]. On the other hand it is finite by 7.1 (a), hence equal to $\{0\}$.

By 7.1 (b) we obtain

$$\omega(A+K_{|(Id-P)E}) = \max(\omega_{ess}(A+K_{|(Id-P)E}), s(A+K_{|(Id-P)E})).$$

Since PE is finite dimensional 7.1 (b) yields

$$\omega_{ess}(A+K_{|(Id-P)E}) = \omega_{ess}(A+K) \leq \omega(A) < 0.$$

Moreover $s(A+K_{|(Id-P)E}) < 0$ since $\sigma(A+K_{|(Id-P)E})\cap\{\lambda: Re\lambda > \omega(A)+\varepsilon\}$ consists of a finite number of poles with negative real part for $0<\varepsilon<-\omega(A)$. This shows $\omega(A+K_{|(Id-P)E}) < 0$.

□

VII.4 Reachability of Growth Bound Zero

We investigate in this section the question (7.11).

Proposition 7.3

Let E, U be Banach spaces, $(A,D(A))$ the generator of a strongly continuous semigroup with $\omega(A)<0$, and take $B\in\mathcal{L}(U,E)$, $C\in\mathcal{L}(E,U)$.

(a) Each of the following conditions implies the existence of $\mu\in\mathbb{C}$ such that $\omega(A+\mu BC) \geq s(A+\mu BC) \geq 0$.

(i) E is finite dimensional and $\sigma(BC) \neq \{0\}$.

(ii) $0\in\rho(BC)$ and $\sigma((BC)^{-1}A) \neq \emptyset$.

(b) Each of the following conditions implies the existence of $\mu>0$ such that $\omega(A+\mu BC) \geq s(A+\mu BC) \geq 0$

(i) E is finite dimensional and $\sigma(BC) \not\subseteq \{\lambda: \mathrm{Re}\lambda\leq 0\}$.

(ii) $0\in\rho(BC)$ and $\sigma((BC)^{-1}A) \cap \{\lambda: \lambda\leq 0\} \neq \emptyset$.

(iii) E is a Banach lattice, $A \in \mathcal{L}(E)$, $T > 0$, $BC > 0$ and $r(BC) > 0$.

Proof: (b)(i): Let $\sigma(BC) \not\subseteq \{\lambda: \mathrm{Re}\lambda\leq 0\}$. Since the spectrum varies continuously under perturbations on finite dimensional spaces (see e.g. [11, II.5.1]) we find $\delta>0$ such that $\sigma(BC+\delta A) \not\subseteq \{\lambda: \mathrm{Re}\lambda\leq 0\}$. Let $\lambda\in\sigma(BC+\delta A)$, $\mathrm{Re}\lambda>0$, then $\omega(A + \frac{1}{\delta}BC) = s(A + \frac{1}{\delta}BC) \geq \frac{\mathrm{Re}\lambda}{\delta} > 0$.

(a)(i): $\sigma(BC) \neq \{0\}$ implies $\sigma(\alpha BC) \not\subseteq \{\lambda: \mathrm{Re}\lambda\leq 0\}$ for some $\alpha\in\mathbb{C}$, hence by (b) (i) $\omega(A+\mu\alpha BC) = s(A+\mu\alpha BC) > 0$ for some $\mu>0$.

(a) (ii) and (b) (ii): Let $-\mu\in\sigma((BC)^{-1}A)$, i.e.

$0 \in \sigma((BC)^{-1}A+\mu Id)$. Then $0 \in \sigma(A+\mu BC)$, since otherwise $(A + \mu BC)^{-1}BC$ would be the inverse of $(BC)^{-1}A + \mu Id$.

(b) (iii): As mentioned in the proof of 5.8 we know by [16, C-II,1.11] that $A + \|A\| \cdot Id > 0$ for a bounded generator of a positive semigroup on a Banach lattice. Let for all $\mu \geq 0$ $D_\mu := A + \|A\| \cdot Id + \frac{\mu}{r(BC)}BC$, then $D_\mu \geq \frac{\mu}{r(BC)}BC \geq 0$ and $\|(D_\mu)^n\| \geq \|(\frac{\mu}{r(BC)}BC)^n\| \geq \mu^n$, hence $\omega(D_\mu) = s(D_\mu) = r(D_\mu) \geq \mu$ [20, V.4.1], i.e. $\omega(A + \frac{\mu}{r(BC)}BC) = s(A + \frac{\mu}{r(BC)}BC) = s(D_\mu) - \|A\| \geq \mu - \|A\|$ for all $\mu \geq 0$ which yields the assertion (iii).

□

The following three examples show that the second conditions in (i) and (ii) cannot be omitted. In particular we see that there is a triple A, B, C with $BC \neq 0$ such that $\omega(A+\mu BC) < 0$ for all $\mu \in \mathbb{C}$.

Example 7.4

(a) Let $E = \mathbb{R}^2$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $BC = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then
$\omega(A+\mu BC) = -1$ for all $\mu \in \mathbb{C}$ if $\lambda = 0$,
$\omega(A+\mu BC) \leq -1$ for all $\mu \geq 0$ if $Re\lambda \leq 0$.

(b) Let T be a nilpotent semigroup (i.e. there exists $t_0 > 0$ such that $T(t) = 0$ for all $t \geq t_0$) and $BC = Id$, then $\omega(A+\mu BC) = s(A+\mu BC) = -\infty$ for all $\mu \in \mathbb{C}$.

(c) If $\omega(A) < 0$ and $BC = -Id$, then $\omega(A+\mu BC) < 0$ for all $\mu > 0$.

Up to now C was fixed. The following proposition shows how $\omega(A+BC) \geq 0$ is possible if we impose no restrictions on the choice of C.

Proposition 7.5

Let E,U be Banach spaces, $(A,D(A))$ the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on E with $\omega(A)<0$, and take $0\neq B\in\mathcal{L}(U,E)$.

(a) There exists $C\in\mathcal{L}(E,U)$ of rank 1 such that $0 \in p\sigma(A+BC)$ with one dimensional eigenspace.
In particular, $\omega(A+BC) \geq s(A+BC) \geq 0$.

(b) If E is a Banach lattice, T positive and $BU\cap E_+ \neq \{0\}$ then C in (a) can be chosen such that BC is positive. If moreover U is a Banach lattice and $B>0$ then C can be chosen positive.

Proof: (a) Let $0 \neq f_o = Bu_o$. Then there exists $f'\in E'$ with $\langle f', R(0,A)f_o\rangle = 1$. We choose $C = f'\otimes u_o$, then $BC = f'\otimes f_o$ and $(A+BC)R(0,A)f_o = 0$. On the other hand for $f\in Ker(A+BC)$ we obtain $f = R(0,A)BCf = \langle f',f\rangle \cdot R(0,A)f_o$, hence $Ker(A+BC)$ is one dimensional.

(b) For $T>0$ we know that $R(0,A)>0$ ([16, C-II,p. 253]), hence f' can be chosen positive if $f_o>0$. $BC = f'\otimes f_o$ is then positive, and C is positive if $u_o\in U_+$.

□

For later applications we mention that f' in the proof of the preceding proposition can be chosen such that $\|f'\| = \|R(0,A)f_o\|^{-1}$. We make the following definition.

Definition 7.6

Let E, U be Banach spaces, $(A,D(A))$ the generator of a strongly continuous semigroup on a Banach space E with $\omega(A)<0$. Take $B\in\mathcal{L}(U,E)$ and $f=Bu\neq 0$. For $f'\in E'$ the operator $C:= f'\otimes u\in\mathcal{L}(E,U)$ is called an f - feedback operator if $\langle f', R(0,A)f\rangle = 1$, $\|f'\| = \|R(0,A)f\|^{-1}$.

Remark 7.7

If, as in 7.5 (b), $\mathcal{T} > 0$ and $B > 0$ then $A(R(0,A)f_0) + B(CR(0,A)f_0) = 0$. This means that $(R(0,A)f_0, CR(0,A)f_0)$ is a positive stationary pair (c.f. section 2).

Since by Proposition 7.1 (c) the map $K \to \omega(A+K)$ is continuous on $\mathcal{C}(E)$ there has to exist $\mu>0$ such that $\omega(A+\mu BC) = 0$ for A, B, C as in Proposition 7.5. Moreover $s(A+\mu BC) = \omega(A+\mu BC)$ by 7.1 (b). Thus we obtain the following theorem.

Theorem 7.8

Let E, U be Banach spaces, $(A,D(A))$ the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on E with $\omega(A)<0$ and take $0\neq B\in\mathcal{L}(U,E)$.

(a) There exists $C\in\mathcal{L}(E,U)$ of rank 1 such that
$s(A+BC) = \omega(A+BC) = 0$.

(b) If E is a Banach lattice, $\mathcal{T}$ positive and $BU\cap E_+ \neq \{0\}$ then C in (a) can be chosen such that BC is positive. If moreover U is a Banach lattice and $B>0$ then C can be chosen positive.

VII.5 Main Results

In the following we consider the questions (7.2) to (7.10) of the second section.

If T is an irreducible positive semigroup on a Banach lattice then a positive answer to (7.2) - (7.9) can be given in the following theorem as a consequence of Theorem 7.8 and Corollary 7.2.

Theorem 7.9

Let E be a Banach lattice, U a Banach space, (A,D(A)) the generator of a positive, irreducible semigroup $T = (T(t))_{t\geq 0}$ on E with $\omega(A)<0$. Take $0\neq B\in\mathcal{L}(U,E)$ and let $BU\cap E_+ \neq \{0\}$.

(a) There exists $C\in\mathcal{L}(E,U)$ of rank 1 such that BC is positive and $\omega(A+BC) = 0$.

For any such C zero is a pole of order one of the resolvent of A+BC with one dimensional eigenspace and strictly positive residue P, and the semigroup $S = (S(t))_{t\geq 0}$ generated by (A+BC,D(A)) is positive and tends in norm to P as $t\to\infty$.

(b) C can be chosen positive if U is a Banach lattice and $B > 0$.

Proof: The existence of C was shown in 7.8. S is positive and irreducible by the perturbation formula in [16, A-II,1.29], and 0 is a pole of the resolvent of A+BC with residue P by 7.2 (a). Moreover, irreducibility of S implies by [16, C-III,3.12] that 0

is a pole of order 1 with one dimensional eigenspace, hence by 7.2 (b) $\sigma(A+BC)\cap i\mathbb{R} = \{0\}$ and $\omega(A+BC_{|(Id-P)E}) < 0$, i.e. $S(t) \to P$ as $t\to\infty$. Finally, P is strictly positive by [20, III.8.5] since S is irreducible.

□

The following theorem gives positive answers to the questions (7.2) - (7.4) in part (a), and moreover to (7.5), (7.6) in (b), and to (7.8), (7.9) in (c).

Theorem 7.10

Let E and U be Banach spaces, (A,D(A)) the generator of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on E with $\|T(t)\| \leq M\cdot e^{-\omega t}$ for all $t\geq 0$ and some $M,\omega > 0$, and let $B\in\mathcal{L}(U,E)$.

Let $\|R(0,A)f_o\| = \frac{M}{\omega}$ for some $f_o\in BU$ with $\|f_o\| = 1$.

(a) There exists an f_o-feedback operator $C\in\mathcal{L}(E,U)$ such that the following assertions hold:

0 is a simple eigenvalue of A+BC,

$\sigma(A+BC)\cap i\mathbb{R}$ consists of poles of order one of the resolvent of A+BC with residues of finite rank.

The semigroup $S = (S(t))_{t\geq 0}$ generated by A+BC is bounded.

(b) If in addition $\|R(\lambda,A)f_o\| < \frac{M}{\omega}$ for all $0\neq\lambda\in i\mathbb{R}$ or if E is a Banach lattice and T, f_o are positive then

$\sigma(A+BC)\cap i\mathbb{R} = \{0\}$, and $S(t)$ tends in norm as $t\to\infty$ to the residue P belonging to $0\in\sigma(A+BC)$ with $PE = \text{span}(R(0,A)f_o)$.

(c) If E is a Banach lattice, T positive and $f_o \in E_+$ then C in (a) can be chosen such that BC is positive. If moreover U is a Banach lattice and $f_o \in E_+ \cap BU_+$ then C can be chosen positive.

Proof: (a) We chose C as in 7.5, 7.6 as f_o-feedback operator such that $BC = f' \otimes f_o$, and moreover such that BC resp. C are positive if the conditions of (c) are fulfilled.

Then 0 is a simple eigenvalue of A+BC with eigenspace spanned by $R(0,A)f_o$; and $\|f'\| = \|R(0,A)f_o\|^{-1} = \frac{\omega}{M}$.

For all $\lambda \in \rho(A)$ with $\langle f', R(\lambda,A)f_o \rangle \neq 1$ one verifies that

$$\lambda \in \rho(A+BC),$$
$$R(\lambda,A+BC)f = \frac{\langle f', R(\lambda,A)f \rangle}{1 - \langle f', R(\lambda,A)f_o \rangle} \cdot R(\lambda,A)f_o + R(\lambda,A)f. \tag{7.14}$$

As we will show now this formula implies

$$\sigma(A+BC) \cap \{\lambda : Re\lambda > 0\} = \emptyset, \tag{7.15}$$

hence $s(A+BC) \leq 0$.

Proof of (7.15): It is known ([16, A-I,1.11]) that $\|R(\lambda,A)\| \leq \frac{M}{Re\lambda + \omega}$ for $Re\lambda > \omega(A)$. Therefore we have $|\langle f', R(\lambda,A)f_o \rangle| \leq \frac{\omega}{M} \cdot \frac{M}{Re\lambda+\omega} < 1$ if $Re\lambda > 0$, hence $\lambda \in \rho(A+BC)$ by (7.14).

On $G := \rho(A) \cap \{\lambda \in \mathbb{C} : \langle f', R(\lambda,A)f_o \rangle \neq 1\}$ we define the complex valued function $\chi : \lambda \to (1 - \langle f', R(\lambda,A)f_o \rangle)^{-1}$. Then for all $\lambda \in G$:

$$R(\lambda,A+BC) = \chi(\lambda) \cdot R(\lambda,A)'f' \otimes R(\lambda,A)f_o + R(\lambda,A), \tag{7.16}$$

where $R(\lambda,A)'$ denotes the adjoint operator of $R(\lambda,A)$. Since the maps $\lambda \to R(\lambda,A)'f'$ and $\lambda \to R(\lambda,A)f_o$ are holomorhic in $\rho(A)$ and

have no zeros, we obtain from (7.16) that the poles in $\rho(A)$ (and their order) of the function χ and of the resolvent of A+BC coincide.

Furthermore the order of a pole λ_o of χ is less or equal to k, if $(\lambda-\lambda_o)^k\chi(\lambda)$ is bounded as $\lambda\to\lambda_o$ in $H:=\{\lambda\in\mathbb{C} : Re\lambda>Re\lambda_o\}$.

Proof: Suppose that $\chi(\lambda) = \sum_{n=-k-m}^{\infty} a_n(\lambda-\lambda_o)^n$ with $a_{-k-m}\neq 0$, $m\geq 1$, while $(\lambda-\lambda_o)^k\chi(\lambda)$ and therefore $(\lambda-\lambda_o)^{k+m-1}\chi(\lambda)$ is bounded as $\lambda\to\lambda_o$ in H. Then also $a_{-k-m}(\lambda-\lambda_o)^{-1} = (\lambda-\lambda_o)^{k+m-1}\chi(\lambda) - \sum_{n=-k-m+1}^{\infty} a_n(\lambda-\lambda_o)^{n+k+m-1}$ is bounded as $\lambda\to\lambda_o$ in H. This contradicts $a_{-k-m}\neq 0$.

After these considerations we are now able to prove the remaining assertions.

By assumption $s(A)\leq\omega(A)<0$. By 7.1 for all $\varepsilon>0$ and $t>0$ the sets $\sigma(S(t))\cap\{\lambda : |\lambda|>e^{(\omega(A)+\varepsilon)t}\}$ and $\sigma(A+BC)\cap\{\lambda: Re\lambda>\omega(A)+\varepsilon\}$ consist of a finite number of poles of the resolvent of S(t), resp. of A+BC with residues of finite rank. In particular the sets $\sigma(S(t))\cap\Gamma$ (Γ denotes the unit circle in $\mathbb{C}$) and $\sigma(A+BC)\cap i\mathbb{R}$ consist of a finite number of poles with residues of finite rank. We show that the order of every pole $\lambda_o\in\sigma(A+BC)\cap i\mathbb{R}$ is 1. As already mentioned $\|R(\lambda,A)\|\leq\frac{M}{Re\lambda+\omega}$ for $Re\lambda>\omega(A)$. By the choice of f' for $Re\lambda\geq\omega(A)$ we therefore have:

$|(\lambda-\lambda_o)\chi(\lambda)| \leq Re\lambda\cdot(1-\|f'\|\frac{M}{Re\lambda+\omega})^{-1} = Re\lambda\cdot(1-\frac{\omega}{Re\lambda+\omega})^{-1} = Re\lambda+\omega$,

hence $(\lambda-\lambda_o)\chi(\lambda)$ is bounded as $\lambda\to\lambda_o$ in $\{\lambda\in\mathbb{C} : Re\lambda>Re\lambda_o=0\}$, and therefore λ_o is a pole of order 1 of χ and of $R(\cdot,A+BC)$.

By [9, 1.10] the sum P of the residues of the resolvent of A+BC on $i\mathbb{R}$ and the sum of the residues of the resolvent of S(t) on Γ

are identical for all $t>0$.

Let $\sigma(A+BC)\cap i\mathbb{R} = \{0,\lambda_1, \dots ,\lambda_n\}$ with residues $P_0,P_1, \dots ,P_n$. Since $P = \sum_{i=0}^{n} P_i$ has finite rank we obtain for all $t>0$:

$$S(t)P = \sum_{i=0}^{n} \exp(t(A+BC)P_i) = P_0 + \sum_{i=1}^{n} \exp(\lambda_i t)P_i,$$

hence $\|S(t)P\| \leq \sum_{i=0}^{n} \|P_i\|$, which shows that $S_{|PE}$ is bounded. For $0<\varepsilon<-\omega(A)$, the set $\sigma_1(t) := (\sigma(S(t))\setminus\Gamma) \cap \{\lambda:|\lambda|>e^{(\omega(A)+\varepsilon)t}\}$ consists of a finite number of poles of the resolvent of $S(t)$. All these poles have absolute value less than 1, since otherwise by [9, 1.10] there would exist a pole of the resolvent of A+BC with real part greater than 0 in contradiction to (7.15).

Since $\sigma(S(t))\setminus\Gamma = \sigma(S(t))_{|(Id-P)E}$ we have $r(S(t)_{|(Id-P)E}) \leq \max(e^{(\omega(A)+\varepsilon)t}, \max\{|\lambda|:\lambda\in\sigma_1(t)\}) < 1$, hence $\omega(A+BC_{|(Id-P)E}) < 0$, i.e. $\|S(t) - S(t)P\| \to 0$ as $t\to\infty$; in particular S is bounded.

(b) Let $\|R(\lambda,A)f_o\| < \frac{M}{\omega}$ for all $\lambda\in i\mathbb{R}\setminus\{0\} \subset \rho(A)$. Then for all such λ: $|\langle f', R(\lambda,A)f_o\rangle| < \|f'\|\cdot\frac{M}{\omega} = 1$, hence $\lambda\in\rho(A+BC)$ by (7.14). Therefore $\sigma(A+BC)\cap i\mathbb{R} = \{0\}$ and $S(t) \to P$ in norm as $t\to\infty$ by what was shown in (a).

If E is a Banach lattice and T, f_o are positive then the assertion (b) follows from Corollary 7.2 (b).

(c) This is true by the choice of C in (a).

□

Our next theorem answers (7.2) - (7.6), (7.8) in part (a) and (b), (7.9) in (c), and (7.10) in (b) and (c).

Theorem 7.11

Let E be a Banach lattice, U a Banach space, $(A,D(A))$ the generator of a positive semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on E with $\omega(A)<0$. Take $B\in\mathcal{L}(U,E)$ and $0\neq f_o\in E_+\cap BU$.

(a) If $R(0,A)f_o$ is a quasi-interior point of E_+, then there exists an f_o-feedback operator $C\in\mathcal{L}(E,U)$ such that the following assertions hold:

BC is positive,

$\sigma(A+BC)\cap i\mathbb{R} = \{0\}$,

0 is a pole of order one with residue P, $PE = \text{span}(R(0,A)f_o)$, and the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $(A+BC,D(A))$ tends in norm to P as $t\to\infty$.

(b) If E (as a Banach lattice) is isomorphic to an AL-space (in particular if E is finite dimensional) then there exists an f_o-feedback operator $C\in\mathcal{L}(E,U)$ with the properties stated in (a).

(c) If U is a Banach lattice and $f_o\in BU_+$ then C in (a) and (b) can be chosen positive.

Proof: We choose C as in the proof of 7.10 and proceed exactly in the same way as in the proof there. We only have to show (7.15) and that every pole of the resolvent of A+BC on $i\mathbb{R}$ has order 1. The other conclusions in the proof of 7.10 are obviously valid for all positive semigroups on Banach lattices.

We show (7.15). Let $\text{Re}\lambda>0$. Then $\mathcal{T}>0$, $f_o>0$ and $f'>0$ imply

$$|\langle f', R(\lambda,A)f_o\rangle| = |\langle f', \int_0^\infty e^{-\lambda t}T(t)f_o dt\rangle| \leq$$

$$\int_0^\infty e^{-\text{Re}\lambda t}\cdot\langle f', T(t)f_o\rangle dt < \int_0^\infty \langle f', T(t)f_o\rangle dt =$$

$$\langle f', R(0,A) f_o \rangle = 1,$$

i.e. $\lambda \in \rho(A+BC)$ by (7.14). This proves (7.15).

Under the assumptions of (a) we know by Proposition 7.1 (b) and (7.15) that $\omega_{ess}(A+BC) = \omega_{ess}(A) < 0 = s(A+BC) = \omega(A+BC)$. Hence by [16, C-III,3.16] $s(A+BC) = 0$ is a strictly dominant eigenvalue, and a pole of order 1 of the resolvent of A+BC since the eigenfunction $R(0,A) f_o$ is a quasi-interior point of E_+.

Now let the assumptions of (b) hold, and endow E with an AL-norm. Then by continuity of T there exists $t>0$ such that

$$\left\| \int_0^t T(s) f_o ds \right\| = \left\| \int_t^\infty T(s) f_o ds \right\| = \frac{1}{2} \left\| \int_0^\infty T(s) f_o ds \right\| = \frac{1}{2} \| R(0,A) f_o \|.$$

This implies that for all λ with $\mathrm{Re}\lambda > 0$:

$$\| R(\lambda,A) f_o \| = \left\| \int_0^\infty e^{-\lambda s} T(s) f_o ds \right\| \leq$$

$$\left\| \int_0^t T(s) f_o ds \right\| + e^{-t \cdot \mathrm{Re}\lambda} \left\| \int_t^\infty T(s) f_o ds \right\| =$$

$$\frac{1}{2}(1 + e^{-t \cdot \mathrm{Re}\lambda}) \| R(0,A) f_o \| .$$

Hence by choice of f' and with χ as in the proof of 7.10 and for every $\lambda_o \in \sigma(A+BC) \cap i\mathbb{R}$:

$$|(\lambda - \lambda_o)\chi(\lambda)| \leq \mathrm{Re}\lambda \cdot (1 - \|f'\| \cdot \|R(\lambda,A) f_o\|)^{-1} \leq$$

$$\mathrm{Re}\lambda \cdot (1 - \tfrac{1}{2}(1 + e^{-t \cdot \mathrm{Re}\lambda}))^{-1} = 2\mathrm{Re}\lambda \cdot (1 - e^{-t \cdot \mathrm{Re}\lambda})^{-1}.$$

Since the right hand side of this inequality tends to $\frac{2}{t}$ as $\mathrm{Re}\lambda \to 0$ in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ we conclude as in the proof of 7.10 (a) that λ_o is a pole of order 1 of the resolvent of A+BC.

□

Our last theorem in this section answers the questions (7.2) - (7.6) in part (a), (b), (c), and (7.8), (7.9) in (d). Contrary to the preceding theorems it requires that the resolvent of A possesses a pole, but on the other hand weak stability is sufficient in part (a) and (b). Also the proof differs from the preceding ones in that respect that we will make use of techniques from linear algebra rather than of the perturbation result 7.1.

Theorem 7.12

Let E and U be Banach spaces, (A,D(A)) the generator of a weakly stable semigroup $T = (T(t))_{t\geq 0}$ on E, and let λ_o be a pole of the resolvent of A with residue P of finite rank. Take $0\neq B\in\mathcal{L}(U,E)$ and $f_o\in Ker(\lambda_o-A)$, $\|f_o\|=1$.

(a) If $f_o\in BU$ then there exists $C\in\mathcal{L}(E,U)$ such that the following assertions hold:

0 is an eigenvalue of A+BC with eigenspace spanned by f_o,

the semigroup $S = (S(t))_{t\geq 0}$ generated by (A+BC,D(A)) tends weakly to a projection R as $t\to\infty$ with $RE = span(f_o)$.

(b) Assume that $0\in\rho(A)$ and that f_o can be extended to a basis $\{f_o,f_1, \ldots ,f_m\}$ of PE such that the projection Q on $span(f_o)$ along $span(f_1, \ldots ,f_m)\oplus(Id-P)E$ commutes with A (i.e $Qf\in D(A)$ and $AQf = QAf$ for all $f\in D(A)$), and that $Qg_o\neq 0$ for some $g_o\in BU$. Then there exists $C\in\mathcal{L}(E,U)$ such that the following assertions hold:

0 is an eigenvalue of A+BC with eigenspace $span(R(0,A)g_o)$ and the semigroup $S = (S(t))_{t\geq 0}$ generated by (A+BC,D(A)) tends weakly to a projection R as $t\to\infty$ with $RE = span(R(0,A)g_o)$.

(c) If T is strongly or uniformly exponentially stable then $\lim_{t\to\infty} S(t)$ in (a) and (b) exists in the strong, resp. norm topology of $L(E)$.

Before proving the theorem we make some remarks.

In (a) the assumption $f_o \in BU$ is quite restrictive. In this respect the assumption $Qg_o \neq 0$ in (b) is much weaker, provided that there exists a basis with the required properties.
If for example λ_o is a pole of order 1 of the resolvent of A then such a basis exists (and consists of eigenvectors to λ_o only). Hence, if the spectrum of A consists of poles of order 1 only and if the eigenvectors form a Schauder basis of E then to every $g_o \in BU$ there is at least one eigenvector f_o with projection Q such that $Qg_o \neq 0$. In this case to every $B \neq 0$ we can find an eigenvector f_o such that the assumption of (b) is fulfilled.
If λ_o is not a pole of order 1 then the assumption in (b) means that the representation of $A_{|PE}$ in Jordan normal form possesses at least one Jordan block of size 1.

Proof of 7.12: (a) First we remark that $Re\lambda < 0$ for every $\lambda \in p\sigma(A)$ since T is weakly stable. Let $Bu_o = f_o$, choose f' such that $\langle f', f_o \rangle = 1$, $f'_{|(Id-P)E} = 0$ and set $C := -\lambda_o f' \otimes u_o$. Then

$$BC_{|(Id-P)E} = 0 \ . \tag{7.17}$$

By the perturbation formula in [16, A-II,1.29] we obtain

$$S(t) = T(t)(Id-P) + S(t)P, \tag{7.18}$$

and PE is invariant under S(t).

Obviously 0 is an eigenvalue of A+BC with eigenvector f_o.

Moreover

$$\begin{aligned}&0 \text{ is a simple eigenvalue of } A+BC \text{ and a pole}\\ &\text{of order 1 of the resolvent of } A+BC_{|PE}.\end{aligned} \tag{7.19}$$

Proof: Let $f\in D(A^n)$ and $(A+BC)^n f = 0$ for some $n\geq 1$. We have to show that $f\in \text{span}(f_o)$.

Now by (7.17)

$$0 = (A+BC)^n f = (A+BC)^n(Id-P)f + (A+BC)^n Pf =$$
$$A^n(Id-P)f + (A+BC)^n Pf,$$

the summands lying in (Id-P)E, resp. PE. Hence

$$0 = A^n(Id-P)f = (A+BC)^n Pf\ . \tag{7.20}$$

Since $0\notin p_\sigma(A)$ we conclude that $(Id-P)f = 0$ and it remains to show that $Pf\in\text{span}(f_o)$. Suppose that this is not true. Then we show per induction that $(A+BC)^m Pf\notin\text{span}(f_o)$ for all $m\in\mathbb{N}$ in contradiction to (7.20). For m=0 the claim is just what we supposed. Assume that it is true for m-1 and set

$$g := (A+BC)^{m-1}Pf\notin\text{span}(f_o)\ .$$

Then

$$(A+BC)^m Pf = (A-\lambda_o)g + \lambda_o g + BCg.$$

Case 1: $(A-\lambda_o)g = 0$. Since $\lambda_o g \notin \text{span}(f_o)$ and $BCg \in \text{span}(f_o)$ the claim for m follows.

Case 2: $(A-\lambda_o)g \neq 0$. Since $g \in PE$ there is $k>1$ such that

$$(A-\lambda_o)^k g = 0 \neq (A-\lambda_o)^{k-1} g$$

and therefore

$$(A-\lambda_o)^{k-1}(A+BC)^m Pf = (A-\lambda_o)^{k-1} Ag + (A-\lambda_o)^{k-1} BCg =$$

$$\lambda_o (A-\lambda_o)^{k-1} g + (A-\lambda_o)^{k-1} BCg.$$

The second summand is 0 since $BCg \in \text{span}(f_o)$, hence

$$(A-\lambda_o)^{k-1}(A+BC)^m Pf = \lambda_o (A-\lambda_o)^{k-1} g \neq 0,$$

i.e. $(A+BC)^m Pf \notin \text{Ker}(A-\lambda_o)$, in particular $(A+BC)^m Pf \notin \text{span}(f_o)$.

Thus we have proven (7.19).

Let now $\tilde{R}$ denote the residue belonging to $0 \in \sigma(A+BC_{|PE})$.

It remains to show that $S(t)$ tends weakly to $R := \tilde{R}P$ as $t \to \infty$.

To this end we claim first that

$$\sigma(A+BC_{|PE}) \subset \{0, \lambda_o\} . \tag{7.21}$$

Proof: Let $(\lambda-A-BC)Pf = 0$, $Pf \neq 0$. Since $\lambda_o \in \sigma(A_{|PE})$ there is $n \geq 1$ such that $(\lambda_o-A)^n Pf = 0 \neq (\lambda_o-A)^{n-1} Pf$.

Case 1: If n=1 then

$$(\lambda-\lambda_o)Pf = (\lambda-A)Pf = BCPf = -\langle f', Pf\rangle \cdot \lambda_o f_o . \tag{7.22}$$

Hence $\lambda = \lambda_o$ or $Pf \in \text{span}(f_o)$. If the latter is the case then

$BCPf = -\lambda_o Pf$ according to the choice of BC, hence $\lambda=0$ by (7.22).

Case 2: If $n\geq 2$ then

$$(\lambda_o - A)^{n-1}(\lambda - A)Pf = -(\lambda_o - A)^{n-1}\cdot\langle f', Pf\rangle\cdot\lambda_o f_o = 0 \ .$$

On the other hand

$$(\lambda_o - A)^{n-1}(\lambda - A)Pf = (\lambda_o - A)^n Pf + (\lambda - \lambda_o)(\lambda_o - A)^{n-1}Pf = (\lambda - \lambda_o)(\lambda_o - A)^{n-1}Pf,$$

hence $\lambda = \lambda_o$.

Thus (7.21) is proven.

(7.21), (7.19) and the spectral mapping theorem imply that

$$\sigma(S(t)_{|PE}) \subset \{1, \exp(\lambda_o t)\}$$

and that

1 is a pole of order 1 of the resolvent of $S(t)_{|PE}$ with residue $\tilde{R}$.

Hence

$$\sigma(S(t)_{|(P-\tilde{R}P)E}) \subset \{\exp(\lambda_o t)\}$$

and

$$S(t)P = S(t)\tilde{R}P + S(t)(P-\tilde{R}P) \to \tilde{R}P = R \text{ as } t\to\infty$$

in the norm topology of $L(E)$. (7.18) and weak stability of T imply that $S(t)$ tends weakly to R as $t\to\infty$.

(b) Since Q has finite rank it is immediate that T(t) and R(0,A) also commute with Q.

Let $Bu_o = g_o$. Without loss of generality we can assume that $R(0,A)Qg_o = QR(0,A)g_o = \|QR(0,A)g_o\|f_o$. Otherwise take $\lambda \cdot g_o$ instead of g_o for a suitable $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Choose f' such that $\langle f', f_o \rangle = 1$, $f'|_{(Id-Q)E} = 0$ and set $C := f' \otimes \|R(0,A)Qg_o\|^{-1} u_o$. Then

$$BC|_{(Id-Q)E} = 0 \,. \tag{7.23}$$

0 is an eigenvalue of A+BC with eigenvector $R(0,A)g_o$ since

$$(A+BC)R(0,A)g_o = -g_o + BCQR(0,A)g_o =$$
$$-g_o + \|QR(0,A)g_o\|BCf_o = 0.$$

Moreover $\operatorname{codim}((Id-Q)E) = 1$ and $QR(0,A)g_o = R(0,A)Qg_o \neq 0$, i.e. $R(0,A)g_o \notin (Id-Q)E$, hence

$$E = \operatorname{span}(R(0,A)g_o) \oplus (Id-Q)E.$$

Let R be the projection onto $\operatorname{span}(R(0,A)g_o)$ belonging to this decomposition of E.

We claim that $S(t) \to R$ weakly as $t \to \infty$ (this proves also that 0 is a simple eigenvalue of A+BC).

By (7.23) and by the perturbation formula in [16, A-II,1.29] T and S coincide on $(Id-Q)E$. Since $(Id-Q)E = (Id-R)E$ we obtain

$$S(t) = S(t)(Id-R) + S(t)R = T(t)(Id-R) + R.$$

Weak stability of T yields the claim.

(c) This is obvious from the proofs of (a) and (b).

□

Remark 7.13

In the proof of 7.12 we showed that the feedback operator C in (a) can be chosen such that $C = -\lambda_o f' \otimes u_o$ where $Bu_o = f_o$, $\langle f', f_o \rangle = 1$, $f'|_{(Id-P)E} = 0$; and in (b) such that $C = \mu f' \otimes u_o$ where $Bu_o = g_o$, $\langle f', f_o \rangle = 1$, $f'|_{(Id-Q)E} = 0$, and μ is the complex number with the property that $\mu R(0,A)Qg_o = f_o$.

VII.6 Remarks

In this section we make some additional remarks on the theorems from the preceding section. Their assertions will be compared by applying them to the standard examples of translation and multiplication semigroups.

Remark 7.14

(a) The assumption in 7.10 not only is a strong requirement on the range of B, but for many semigroups impossible to achieve (see Examples 7.15 and 7.16).

Since $\|R(0,A)f_o\| = \|\int_0^\infty T(t)f_o dt\| \leq \int_0^\infty \|T(t)f_o\| dt \leq \int_0^\infty M \cdot e^{-\omega t} dt = \frac{M}{\omega}$ the assumption implies $\|T(t)f_o\| = M \cdot e^{-\omega t}$ for all $t>0$. Since $\|T(0)f_o\| = 1$ we infer $M=1$ and $\|T(t)f_o\| = \|T(t)\| = e^{-\omega t}$, hence $\omega(A) = -\omega$.

(b) The assumption in 7.10 (a) is fulfilled e.g. if $Af_o = -\omega f_o$; in this case we have $R(0,A)f_o = \frac{1}{\omega} f_o$, hence $\|R(0,A)f_o\| = \frac{1}{\omega}$. In

particular this is the case if $-\omega$ is a pole with eigenvector f_o, yet we then obtain a better result with 7.12 (a) (if the residue has finite rank). It is interesting that in this case 7.10 and 7.12 in general yield a different C, as the Examples 7.18 and 7.19 in the next section will show.

(c) As shown in (a) 7.10 implies that T is a contraction semigroup. By means of the theorem of Lumer-Phillips we can prove some of the assertions of 7.10 under more general assumptions in a simpler way:

Let E and U be Banach spaces, (A,D(A)) the generator of a strongly continuous semigroup with $\|T(t)\| \leq e^{-\omega t}$ for all $t \geq 0$ and some $\omega > 0$, and let $B \in \mathcal{L}(U,E)$.

If for some $f_o \in BU$ with $\|f_o\| = 1$ and for an f_o-feedback operator $C = f' \otimes u_o$ there exists a duality map j (for definition see [25, XIV.6]) with $\langle f', f \rangle \cdot \langle j(f), f_o \rangle - \omega \|f\|^2 \leq 0$ for all $f \in D(A)$ (in 7.10 this is fulfilled by every duality map), then (A+BC,D(A)) generates a contraction semigroup and $0 \in p_\sigma(A+BC)$.

Proof: For all $f \in D(A)$ the following holds

$$\langle j(f), (A+BC)f \rangle = \langle j(f), Af \rangle + \langle f', f \rangle \cdot \langle j(f), f_o \rangle \leq$$
$$\langle j(f), Af \rangle + \omega \|f\|^2 = \langle j(f), (A+\omega)f \rangle \leq 0$$

since $A+\omega Id$ generates a contraction semigroup by assumption. The assertion follows from the Lumer-Phillips theorem ([25, IX.8]).

(d) The examples 7.15 and 7.16 below show that in many instances the assertions of the theorems 7.10 and 7.11 hold for an f_o-feed-

back operator C even if none of the assumptions in 7.10 and 7.11 are fulfilled for any $f_o \in BU$. We even do not know any counter-example where the assertions do not hold for an f_o-feedback operator. In fact in many examples it is rather simple by means of the formula (7.14) to show that 0 is a pole of order 1 of the resolvent of A+BC with one dimensional eigenspace and a strictly dominant eigenvalue; then the assertions of 7.10 and 7.11 follow from 7.1 (b).

Example 7.15

Let $\mathcal{T}_1 = (T_1(t))_{t\geq 0}$ and $\mathcal{T}_2 = (T_2(t))_{t\geq 0}$ be the modified translation semigroups on $E := C_o(\mathbb{R}_+)$ given by

$$T_1(t)f(x) = e^{-t}f(x+t) \quad \text{for } x,t\geq 0,$$

$$T_2(t)f(x) = e^{-t}\begin{cases} f(x-t) & \text{for } x\geq t\geq 0 \\ f(0) & \text{for } t>x\geq 0. \end{cases}$$

Let U be a Banach space and $B\in\mathcal{L}(U,E)$.

For i=1,2 the generators $(A_i,D(A_i))$ of $\mathcal{T}_i$ are given by

$$A_1f = f'-f, \quad D(A_1) = \{f\in E : f\in C^1, A_1f\in E\},$$
$$A_2f = -f'-f, \quad D(A_2) = \{f\in E : f\in C^1, A_2f\in E, f'(0)=0\}.$$

We know that $s(A_i) = \omega(A_i) = -1$ for i=1,2, and for $Re\lambda\geq -1$ the resolvent of A_i are given by

$R(\lambda,A_i)f(x) = \int_0^\infty e^{-\lambda t}(T_i(t)f)(x)dt$ for $x\geq 0$, hence

$$R(\lambda,A_1)f(x) = e^{(\lambda+1)x}\cdot\int_x^\infty e^{(-\lambda-1)t}f(t)dt\,, \tag{7.24}$$

$$R(\lambda,A_2)f(x) = e^{(-\lambda-1)x}\left[\int_0^x e^{(\lambda+1)t}f(t)dt + \frac{f(0)}{\lambda+1}\right]. \tag{7.25}$$

Since T_i is not irreducible we cannot apply Theorem 7.9.

(a) As mentioned in Remark 7.14 (a) the assumption of 7.10 implies that $\|T_i(t)f\| = \|T_i(t)\| = e^{-t}$ for some $f \in BU$ with $\|f\|=1$. Obviously this is impossible for i=1, but for i=2 this holds even for all $f \in E$. In fact we have for $\|f\|=1$

$$|R(0,A_2)f(x)| \leq e^{-x}\left|\int_0^x e^t f(t)dt + f(0)\right|$$

$$\leq e^{-x}((e^x-1)\|f\| + \|f\|) = 1,$$

and this estimate shows that $|R(0,A_2)f(x)| = 1$ if and only if $f(t) = c$ for all $t \in [0,x]$ and some $c \in \mathbb{C}$ with $|c|=1$. Hence the assumption of 7.10 is fulfilled for all f_o with $\|f_o\| = |f_o(0)| = 1$. Moreover we have for all those f_o and $0 \neq \lambda \in i\mathbb{R}$

$$|R(\lambda,A_2)f(x)| \leq e^{-x}\left|\int_0^x e^{(\lambda+1)t} f(t)dt + \frac{f(0)}{\lambda+1}\right| \leq$$

$$e^{-x}\left(\int_0^x e^t\|f\|dt + \frac{1}{|\lambda+1|}\right) < 1,$$

and the assumption of 7.10 (b) is also fulfilled.

(b) We consider the assumption of 7.11 (a). For $0 < f_o \in E$ the ideal generated by $R(0,A_i)f_o$ is dense in E if and only if

$$R(0,A_i)f_o(x) > 0 \text{ for all } x \geq 0.$$

For i=1 we see by (7.24) that this holds if and only if f_o does not vanish on any interval $[x,\infty)$.
For i=2 it holds by (7.25) if and only if $f_o(0) > 0$.

(c) We now show that even if for some fixed $f_o = Bu_o$ none of the assumptions of 7.10, 7.11 is fulfilled the assertions of these theorems may still hold for an f_o-feedback operator.

Take e.g. $f_o = 1_{[0,1]} > 0$; then we see by (a) and (b) that the assumptions of 7.10 and 7.11 (a) are not fulfilled for i=1.

But we have

$$R(\lambda,A_1)f_o(x) = \frac{1}{\lambda+1}\begin{cases}1 - \exp(\lambda+1)(x-1) & \text{for } 0\leq x\leq 1\\ 0 & \text{for } x>1\end{cases},$$

hence $\|R(0,A_1)f_o\| = R(0,A_1)f_o(0) = 1 - e^{-1}$ and an f_o-feedback operator is given by

$$C = (1-e^{-1})^{-1}\delta(0)\otimes u_o,$$

where $\delta(0)$ denotes the Dirac-measure in 0.

By (7.14) we obtain

$$R(\lambda,A+BC)f = \frac{R(\lambda,A_1)f(0)}{(1-e^{-1})\left(1-\frac{(1-\exp(-\lambda-1))}{(1-\exp(-1))(\lambda+1)}\right)}R(\lambda,A_1)f_o + R(\lambda,A_1)f\ .$$

Since

$$\left[(1-\exp(-1)) - \frac{(1-\exp(-\lambda-1))}{\lambda+1}\right]^{-1} =$$

$$\frac{\lambda+1}{\lambda(1-\exp(-1))-\exp(-1)+\exp(-\lambda-1)} =$$

$$\frac{\lambda+1}{\lambda}\left[(1-\exp(-1) + \frac{\exp(-\lambda-1)-\exp(-1)}{\lambda}\right]^{-1},$$

we see that $R(\lambda,A_1+BC)$ has a pole of order 1 in $\lambda=0$.

By the formula it is also clear that $s(A+BC) = 0$ (otherwise we would obtain $(1-\exp(-\lambda-1)) = (\lambda+1)(1-\exp(-1))$ for some $\lambda>0$, hence

$(1-\exp(-\lambda-1)) < 1-\exp(-1)$, $\exp(-\lambda-1) > \exp(-1)$ which is not true).

By the 7.2 (b), 7.5 and 7.6 we obtain all the assertions of 7.11.

(d) Similarly it can be shown that for i=1 and for the function $f_o : x \to e^{-x}\sin(x)\cdot[\sup_{y\geq 0}(e^{-y}\sin(y))]^{-1}$ the assertions of 7.10 (b) hold for an f_o-feedback operator, although f_o is not a positive function and hence the assumptions of 7.10, 7.11 are not fulfilled.

□

Example 7.16

Let $T = (T(t))_{t\geq 0}$ be a multiplication semigroup on $E = L^p(\Omega)$, $\Omega\subset\mathbb{R}^n$, $1\leq p<\infty$ given by

$$T(t)f(x) = \exp(t\cdot g(x))\cdot f(x),$$

for some measurable function g on Ω with $\operatorname{ess\,sup}_{x\in\Omega}\operatorname{Re}(g(x)) := \inf\{\alpha\in\mathbb{R} : \operatorname{Re}(g(x)) < \alpha \text{ almost everywhere on } \Omega\} < 0$.

Let U be a Banach space and $B\in\mathcal{L}(U,E)$.

The generator of the semigroup is given by

$$Af(x) = g(x)\cdot f(x),\ D(A) = \{f\in E:\ g\cdot f\in E\},$$

and the following holds

$$s(A) = \omega(A) = \operatorname{ess\,sup}_{x\in\Omega}\operatorname{Re}(g(x)),$$

$$\|T(t)\| \leq e^{\omega(A)t} \text{ for all } t\geq 0,$$

$$R(\lambda,A)f(x) = \frac{f(x)}{\lambda-g(x)} \text{ for } \operatorname{Re}\lambda>\omega(A).$$

T is not irreducible (this can be seen from the explicit characterization of ideals in L^p mentioned in 3.11). Thus we cannot apply 7.9. In general the resolvent of A has no poles, so that we cannot apply 7.12. Again we investigate when the assumptions of 7.10 and 7.11 hold.

(a) Let $f_o \in E$, $\|f_o\| = 1$. The assumptions of 7.10 read as $(-\omega(A))^{-p} = \|R(0,A)f_o\|^p = \int_\Omega |g(x)|^{-p} \cdot |f_o(x)|^p dx$. Since $|g(x)| \geq -\mathrm{Re}(g(x)) \geq -\omega(A)$ for almost all $x \in \Omega$ we see that the assertion is equivalent to

$$g(x) = \omega(A) \text{ almost everywhere in } \Omega' := \{x \in \Omega : f_o(x) \neq 0\} .$$

In particular in this case f_o is an eigenvector to $\omega(A)$; moreover the assumption of 7.10 (b) is fulfilled since for $\lambda \in i\mathbb{R} \setminus \{0\}$ we obtain $\|R(\lambda,A)f_o\|^p = \int_{\Omega'} |\lambda - \omega(A)|^{-p} \cdot |f_o(x)|^p dx < \int_{\Omega'} (-\omega(A))^{-p} \cdot |f_o(x)|^p dx = (-\omega(A))^{-p}$.

(b) Obviously T is a positive semigroup if only if g is real almost everywhere in Ω. Let $f_o > 0$. Then the assumptions of 7.11 (a) mean that $f_o(x)(-g(x))^{-1} > 0$ for almost all $x \in \Omega$, i.e.

$$f_o(x) > 0 \text{ for almost all } x \in \Omega .$$

Thus we have found a condition on f_o completely different from the one in (a).

(c) Again we can show that the assertions of the theorems 7.10, 7.11 still hold for an f_o-feedback operator while f_o does not satisfy the above conditions. Take e.g. $\Omega = \mathbb{R}$, $f_o = 1_{[0,1]} = Bu_o$,

and g real with $\operatorname{ess\,sup}_{x\in\mathbb{R}} \mathrm{Re}(g(x)) < 0$. Then it is easy to check that an f_o-feedback operator is given by $C = h \otimes u_o$, where

$$h := [\int_0^1 (-g(x))^{-p} dx]^{-1} \cdot (-g)^{1-p} \cdot 1_{[0,1]}$$

is an L^q-function $(p^{-1}+q^{-1}=1)$.

By (7.14) we obtain for every $f \in E$

$$R(\lambda, A+BC) f =$$

$$\frac{\langle h, R(\lambda,A) f\rangle}{1 - [\int_0^1 (-g(x))^{-p} dx]^{-1} \cdot \int_0^1 (-g(x))^{1-p} (\lambda - g(x))^{-1} dx} R(\lambda,A) f_o +$$

$$R(\lambda,A) f,$$

unless the denominator is zero.

Since

$$[(\int_0^1 (-g(x))^{-p} dx) - (\int_0^1 (-g(x))^{1-p} (\lambda - g(x))^{-1} dx)]^{-1} =$$

$$[(\int_0^1 \frac{\lambda - g(x)}{(-g(x))^{p} (\lambda - g(x))} dx) - (\int_0^1 \frac{-g(x)}{(-g(x))^{p} (\lambda - g(x))} dx)]^{-1} =$$

$$[\lambda \int_0^1 (-g(x))^{-p} (\lambda - g(x))^{-1} dx]^{-1},$$

we see that $R(\cdot, A+BC)$ has a pole of order 1 in 0 and no other poles in $\{\lambda : \mathrm{Re}\lambda \geq 0\}$. By 7.2 (b), 7.5 and 7.6 we obtain the assertions of 7.11.

VII.7 Applications

Example 7.17

We consider the example of the electrically heated oven in II.1. The semigroup T there is stable if and only if $\det A > 0$ (see Example 6.10) what we assume in the following.

T is irreducible if and only if $\beta \cdot \gamma \neq 0$. In this case we can apply 7.9. In view of 7.11 (b) the operator C in 7.9 can be chosen as an f_0-feedback operator, which is by Remark 7.7 and Example 6.10 necessarily equal to $f' \theta 1$, where

$$\langle f', \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \rangle = \det A, \quad \| f' \| = \det A \cdot \left\| \begin{pmatrix} \delta \\ \gamma \end{pmatrix} \right\|^{-1}.$$

This means

$$f' = (\gamma^2+\delta^2)^{-1} \cdot \det A \cdot \begin{pmatrix} \delta \\ \gamma \end{pmatrix},$$

$$A + BC = \begin{pmatrix} -\alpha + (\gamma^2+\delta^2)^{-1} \cdot \det A \cdot \delta & \beta + (\gamma^2+\delta^2)^{-1} \cdot \det A \cdot \gamma \\ \gamma & -\delta \end{pmatrix}.$$

Indeed $\omega(A+BC) = \det(A+BC) = 0$ and all the assertions of 7.9 hold.

If T is not irreducible, i.e. $\beta = 0$ or $\gamma = 0$, then we can apply Theorem 7.11 (b) and again obtain C as above. By [20, III.8.5] the projection P of 7.11 is strictly positive if and only if the semigroup S generated by A+BC is irreducible, which is obviously equivalent to $\gamma \neq 0$.

Thus we have obtained positive answers to the questions (7.2) - (7.10) if $\gamma \neq 0$, and to (7.2) - (7.6), (7.8) - (7.10) if $\gamma = 0$.

Our next example will be a non-positive system.

Example 7.18

We consider a controlled linear damped oscillator system

$$x''(t) + a\cdot x'(t) + b\cdot x(t) - u(t) = 0, \quad x(t),u(t) \in \mathbb{R},$$

where $a,b > 0$. If we set $f(t) = \begin{pmatrix} x'(t) \\ x(t) \end{pmatrix}$ we obtain the two dimensional system

$$\frac{d}{dt}f(t) = Af(t) + Bu(t), \text{ where}$$

$$A := \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The semigroup $T = (T(t))_{t\geq 0}$ generated by A is not positive, and we cannot apply 7.9, 7.11.

The eigenvalues of A are

$$\lambda_{1,2} = -\frac{a}{2} \pm \left(\frac{a^2}{4} - b\right)^{1/2},$$

hence T is stable with $\omega(A) = \lambda_1$.

The assumptions of 7.10 are not fulfilled: Assume that they hold. Then by Remark 7.14 (a) $\exp(-\omega(A))\cdot T(t)$ is a contraction semigroup, and by the theorem of Lumer-Phillips $A-\omega(A)$ is dissipative, i.e. $\langle f,(A-\omega(A))f\rangle \leq 0$ for all $f\in\mathbb{R}^2$. But this is not true for $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Nevertheless, an f_o-feedback operator C still yields positive answers to the questions (7.2) - (7.6) as we show now.

For $f_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in BU$ we have $C = f'\otimes 1$, where $\langle f', R(0,A)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle = 1$,

$\|f'\| = \|R(0,A)\binom{1}{0}\|^{-1}$. Since $R(0,A)\binom{1}{0} = \binom{0}{1/b}$, we obtain

$$C = (0,b),\quad A+BC = \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix},$$

$$\exp(t(A+BC)) = \begin{pmatrix} e^{-ta} & 0 \\ \frac{1}{a}(1-e^{-ta}) & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \frac{1}{a} & 1 \end{pmatrix} \text{ as } t\to\infty.$$

The spectral bound $s(A)$ is always a pole of order 1 of the resolvent of A unless $b = \frac{a^2}{4}$, and therefore we also can apply Theorem 7.12 (b). We do this for

$$b = \frac{3}{4},\ a = 2.$$

Then $\lambda_{1,2} = -1 \pm \frac{1}{2}$ with eigenvectors

$$f_1 = 5^{-1/2}\binom{-1}{2},\ f_2 = \left(\frac{9}{13}\right)^{1/2}\binom{1}{-2/3},$$

and we evaluate the projection Q of 7.12 (b) as

$$Q = \begin{pmatrix} -1/2 & -3/4 \\ 1 & 3/2 \end{pmatrix}.$$

By 7.13 we have $C = f'\circledast\|R(0,A)\binom{-1/2}{1}\|^{-1}$, where $\langle f', f_1\rangle = 1$, $\langle f', f_2\rangle = 0$, hence by 7.13

$$C = 5^{1/2}\left(\frac{1}{2},\frac{3}{4}\right)\cdot\|\binom{-1}{2}\|^{-1} = \left(\frac{1}{2},\frac{3}{4}\right),\quad A+BC = \begin{pmatrix} -3/2 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\exp(t(A+BC)) = \begin{pmatrix} \exp(-\frac{3}{2}t) & 0 \\ \frac{2}{3}(1-\exp(-\frac{3}{2}t)) & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}.$$

Notice that we have obtained an operator C different from the f_o-feedback operator above (see also Remark 7.14 (b)).

Example 7.19

We consider the example of II.2. As shown there and in 3.11 the semigroup T is uniformly exponentially stable and irreducible, and we can apply 7.9. Let e.g. $\psi \equiv 1$, then $f_1 : x \to \frac{x-x^2}{2}$ belongs to the positive stationary pair $(f_1,1)$. Since the ideal generated by f_1 is dense in $L^2[0,1]$, we can choose C by 7.11 (a) as an f_1-feedback operator, i.e. $Cf_1 = 1$, $\|C\| = \|f_1\|^{-1} = 2 \cdot 30^{1/2}$,

$$C : f \to 120^{-1} \langle f, f_1 \rangle .$$

Since the resolvent of A contains poles of order 1 we can also apply 7.12 (b) to obtain a different C. The projection Q to the pole $\lambda = -n^2\pi^2$ of $R(\cdot,A)$ in 7.12 (b) is given by $Qf = \langle f, e_n \rangle e_n$. Let again $\psi \equiv 1$, then

$$QB1 = \langle \psi, e_n \rangle e_n = e_n \cdot \begin{cases} 2\sqrt{2}/n\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence all odd n are "good" to design a C. Since

$$\|R(0,A)QB1\| = \|n^{-2}\pi^{-2}\langle \psi, e_n \rangle e_n\| = 2\sqrt{2}/(n^3\pi^3)$$

we obtain by 7.13

$$C : f \to (n^3\pi^3/(2\sqrt{2}))\langle f, e_n \rangle .$$

Example 7.20

We consider the linear transport equation of Example 3.13 and assume that $\omega(A) < 0$. If $s(A) > -\lambda^*$ then, as mentioned in 3.13, $s(A) = \omega(A)$ is a strictly dominant eigenvalue and pole of the

resolvent of A with residue P. Then we can apply 7.12 (a) provided that $BU \cap PE \neq \{0\}$ which is a strong requirement on B.

If moreover the semigroup T is irreducible and therefore $s(A)$ a pole of order 1, then we can apply 7.9; and also 7.12 (b) if $PBU \neq \{0\}$.

However, the semigroup may neither be irreducible nor $s(A) > -\lambda^*$ may hold. In this case we can still apply 7.11 (b) since our state space is L^1.

APPENDIX

The following five lemmas on order ideals are needed in the fifth chapter. We recall the definition of the ideals I_f , $J_{f'}$ in chapter I.

Lemma A.1

Let C be a compact subset of the positive cone E_+ of a Banach lattice E and $0<f'\in E'$ a positive linear functional.
If there exists $0<f\in cl(coconeC)\cap J_{f'}$, then $C\cap J_{f'} \neq \emptyset$.

Proof: The assertion is trivial if $0\in C$. Therefore let $0\notin C$ and

$$f = \lim_{m\to\infty}\sum_{i=1}^{n(m)}\alpha_{i,m}\cdot f_{i,m} \text{ for some } \alpha_{i,m}>0,\ f_{i,m}\in C,\ n(m)\in\mathbb{N}.$$

Set

$$\beta_{i,m}:= \alpha_{i,m}\|f_{i,m}\| > 0,\ g_{i,m}:= f_{i,m}\cdot\|f_{i,m}\|^{-1}$$

and obtain

$$0 = \langle f',f\rangle = \lim_{m\to\infty}\sum_{i=1}^{n(m)}\alpha_{i,m}\langle f',f_{i,m}\rangle$$

$$= \lim_{m\to\infty}\sum_{i=1}^{n(m)}\beta_{i,m}\langle f',g_{i,m}\rangle .$$

Moreover

$$\sum_{i=1}^{n(m)}\beta_{i,m} = \sum_{i=1}^{n(m)}\beta_{i,m}\|g_{i,m}\| \geq \|\sum_{i=1}^{n(m)}\beta_{i,m}\cdot g_{i,m}\| \to \|f\| \text{ as } t\to\infty.$$

Thus we can assume that

$\sum_{i=1}^{n(m)} \beta_{i,m} \langle f', g_{i,m} \rangle < \frac{\|f\|}{2m}$ and $\sum_{i=1}^{n(m)} \beta_{i,m} \geq \frac{\|f\|}{2}$ for every $m \in \mathbb{N}$.

Hence for every m there exists i(m) such that $\langle f', g_{i(m),m} \rangle < \frac{1}{m}$. Since C is compact we can find a subsequence of $(f_{i(m),m})_m$ which converges to some $f_o \in C$. Hence a subsequence $(h_m)_m$ of $(g_{i(m),m})_m$ converges to $g := f_o \cdot \|f_o\|^{-1}$, and $\langle f', g \rangle = \lim_{m \to \infty} \langle f', h_m \rangle = 0$. Thus we obtain $\langle f', f_o \rangle = 0$, $f_o \in J_{f'}$.

□

Lemma A.2

Let the assumptions be as in Definition 5.2, $U = \mathbb{R}^m$, e(1), ... , e(m) the canonical unit vectors of $\mathbb{R}^m$ and $0 < f' \in E'$. Then the following assertions are valid.

(a) If $0 < f \in cl(R_t^+) \cap J_{f'}$, then

$\{T(s)Be(i) : 0 \leq s \leq t, 1 \leq i \leq m, Be(i) \neq 0\} \cap J_{f'} \neq \emptyset$.

(b) Assume that $0 \notin \{T(s)Be(i) : 0 \leq s, 1 \leq i \leq m, Be(i) \neq 0\}$, and that $\lim_{t \to \infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|}$ exists for every $1 \leq i \leq m$ for which $Be(i) \neq 0$. If $0 < f \in cl(R^+) \cap J_{f'}$, then

$[\{T(s)Be(i) : 0 \leq s, 1 \leq i \leq m, Be(i) \neq 0\} \cup$

$\{\lim_{t \to \infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|} : 1 \leq i \leq m, Be(i) \neq 0\}] \cap J_{f'} \neq \emptyset$.

Proof: We can assume that $Be(i) \neq 0$ for all $1 \leq i \leq m$; otherwise we restrict B to the sublattice of $\mathbb{R}^m$ spanned by those vectors e(i) for which $Be(i) \neq 0$. This will not affect the set R_t^+, and the assumptions of the lemma remain valid.

(a) $\{T(s)Be(i) : 0 \leq s \leq t\}$ is compact for $1 \leq i \leq m$, since T is strongly continuous. Set $C := \{T(s)Be(i) : 0 \leq s \leq t, 1 \leq i \leq m\}$ and apply

Proposition 5.6 (b) and Lemma A.1 to obtain the assertion.

(b) By Proposition 5.6 (b) we know that the set

$$\text{cocone}[\{\frac{T(s)Be(i)}{\|T(s)Be(i)\|} : 0\leq s,\ 1\leq i\leq m\} \cup \{\lim_{t\to\infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|} : 1\leq i\leq m\}]$$

is dense in R^+. Now continuity of $t \to \frac{T(t)Be(i)}{\|T(t)Be(i)\|}$ and existence of $\lim_{t\to\infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|}$ imply compactness of the set

$$\{\frac{T(s)Be(i)}{\|T(s)Be(i)\|} : 0\leq s,\ 1\leq i\leq m\} \cup \{\lim_{t\to\infty} \frac{T(t)Be(i)}{\|T(t)Be(i)\|} : 1\leq i\leq m\}.$$

By Lemma A.1 we therefore know that this set has non-void intersection with $J_{f'}$. The assertion follows immediately since $\frac{T(t)Be(i)}{\|T(t)Be(i)\|} \in J_{f'}$ if and only if $T(t)Be(i) \in J_{f'}$.

□

Lemma A.3

Let E be a Banach lattice, C a compact subset of E_+, $0 \notin C$ and $(I_i)_{i\in J}$ a class of proper ideals of E which are pairwise orthogonal. Then $C \cap I_i = \emptyset$ for almost all $i \in J$ (i.e. for all except finitely many).

Proof: Assume that C has non-trivial intersection with infinitely many ideals I_i. Since $0 \notin C$ there exists a sequence of pairwise orthogonal elements f_n with $\inf_{n\in\mathbb{N}}(\|f_n\|) > \alpha > 0$. But this implies by [20, II.1.4] that $\|f_n - f_m\| > \alpha$ for all $n \neq m$ in contradiction to the compactness of C.

□

I owe the proofs of the following two lemmas to F. Räbiger.

Lemma A.4

Let E be an infinite dimensional Banach lattice. Then there exists an infinite number of pairwise orthogonal ideals of the form $I_f := \{g \in E : |g| \leq nf,\ n \in \mathbb{N}\}$, $0 < f \in E$.

Proof: First we show the assertion for spaces C(X), X compact and non-finite. We claim that X contains infinitely many pairwise disjoint non-void open subsets O_1, O_2, ... with the property that the complement of $\cup_{i=1}^{n} O_i$ contains an open infinite set. We construct this sequence of sets inductively.

n=1: Let x,y be different elements of X. They can be separated by disjoint open neighborhoods O(x), O(y) of x resp. y. If one of them is an infinite set, say O(x), then take $O_1 = O(y)$, and the complement of O_1 contains O(x). If both are finite sets, then O(y) is closed, hence the complement of it is an infinite and open set and we again take $O_1 = O(y)$.

n→n+1: Assume that we already have pairwise disjoint non-void open sets O_1, ... ,O_n such that the complement of $\cup_{i=1}^{n} O_i$ contains an open infinite set O. Let x,y be different elements in O. Again we can separate them by disjoint open neighborhoods O(x), O(y) of x resp. y. Furthermore we can assume that both O(x) and O(y) are contained in O, since O is open. If one of them is an infinite set, say O(x), then we take $O_{n+1} = O(y)$, and the complement of $\cup_{i=1}^{n+1} O_i$ contains O(x). If both are finite sets, then O(y)

is closed, hence $O \setminus O(y)$ is open and an infinite set. We again take $O_{n+1} = O(y)$ and the complement of $\bigcup_{i=1}^{n+1} O_i$ contains $O \setminus O(y)$. Thus we have proven that E contains infinitely many pairwise disjoint non-void open subsets $O_1, O_2, \ldots$. Let now $x_i \in O_i$ for all $i \in \mathbb{N}$. Then by Urysohn's Lemma there exists for every $i \in \mathbb{N}$ a function $f_i \in C(X)$ such that $f_i(X) \subset [0,1]$, $f_i(x_i) = 1$, $f_i(X \setminus O_i) = \{0\}$. Then $I_{f_i} \neq \{0\}$ for all i; and for $g \in I_{f_i} \cap I_{f_j}$, $i \neq j$ we have $g(x) = 0$ for all $x \in (X \setminus O_i) \cup (X \setminus O_j) = X$, hence $I_{f_i} \cap I_{f_j} = \{0\}$.

Thus we have shown our lemma for spaces C(X). Let now E be an arbitrary infinite dimensional Banach lattice. Then there exists a sequence $f_1, f_2, \ldots$ of linear independent elements of norm 1 in E. For $f := \sum_{i=1}^{\infty} 2^{-i} |f_i|$ the ideal I_f with the canonical gauge functional p_f as norm is a Banach lattice isomorhic to some C(X), X compact ([20, II.7.2 and II.7.4]). Furthermore $f_i \in I_f$ for all $i \in \mathbb{N}$, hence I_f is infinite dimensional. By what was shown above we can find in (I_f, p_f) infinitely many pairwise orthogonal ideals of the form $I_{g_i} := \{g : |g| \leq n g_i, n \in \mathbb{N}\}$, $g_1, g_2, \ldots \in I_f$. Let j be the canonical injection from I_f into E, then $j(I_{g_i}) = \{h \in E : |h| \leq n j(g_i), n \in \mathbb{N}\}$, hence $j(I_{g_i})$, $i = 1,2,\ldots$ are infinitely many pairwise orthogonal ideals of the required form.

□

Lemma A.5

Let E be an infinite dimensional Banach lattice. Then the following assertions are equivalent.

(i) There exists an infinite number of pairwise orthogonal ideals of the form $J_{f'} := \{f \in E : \langle f', |f| \rangle = 0\}$, $J_{f'} \neq \{0\}$, $0 < f' \in E'$.

(ii) There are two orthogonal ideals of the form $J_{f'}$, $J_{f'} \neq \{0\}$, $0 < f' \in E'$.

(iii) There exists a strictly positive linear form f' on E.

Proof: (i)→(ii) is trivial.

(ii)→(iii): Let $J_{f'} \cap J_{g'} = \{0\}$, $0 < f', g' \in E'$ and $0 \leq f \in E$. If $\langle f'+g', f \rangle = 0$, then $0 \leq \langle f', f \rangle \leq \langle f'+g', f \rangle = 0$ and $0 \leq \langle g', f \rangle \leq \langle f'+g', f \rangle = 0$, hence $f \in J_{f'} \cap J_{g'} = \{0\}$. Thus we have shown that f'+g' is strictly positive.

(iii) → (i): Let f' be a strictly positive linear form on E. By Lemma A.4 there exists a sequence of pairwise orthogonal elements f_n in E. Define

$$N(f_n) := \{g' \in E' : \langle |g'|, f_n \rangle = 0\} \quad \text{for all } n \in \mathbb{N}.$$

Since E' is order complete (see I.1), and since $f_n \in E$ is a order continuous linear form on E' ([20, II.4.2]) we can represent E' as the direct sum of $N(f_n)$ and $N(f_n)^{\perp} :=$ $\{g' \in E' : \inf(|g'|, |f'|) = 0 \text{ for all } f' \in N(f_n)\}$ ([20, II.4.10]). Hence $N(f_n)$ is a projection band in E' with band projection $P_n \geq 0$.

Let

$$g_n' := P_n f'.$$

We show that the ideals $I_{g_n'}$ are pairwise orthogonal and different from $\{0\}$.

Since $f'-P_n f' \in N(f_n)^\perp$ we know by [1, 3.10] that $f'-P_n f'$ and $f'-P_m f'$ are orthogonal for $n \neq m$, hence $f'-P_n f' + f'-P_m f' = \sup(f'-P_n f', f'-P_m f') \leq f'$ by [20, II.1.4]. Thus for $n \neq m$ we obtain

$$g_n' + g_m' = 2f' - (f'-P_n f' + f'-P_m f') \geq f' >> 0 .$$

Let now $f \in I_{g_n'} \cap I_{g_m'}$. Then $0 = \langle g_n'+g_m', |f| \rangle$, hence $|f| = 0$ and $f = 0$ since $g_n' + g_m'$ is strictly positive. This shows that the ideals $I_{g_n'}$ are pairwise orthogonal.

They are different from $\{0\}$ since $g_n' \in P_n E' = N(f_n)$ and therefore $f_n \in I_{g_n'}$.

□

REFERENCES

[1] Aliprantis, C.D.; Burkinshaw, O.: Locally Solid Riesz Spaces. New York, San Fransisco, London: Academic Press 1978.

[2] Balakrishnan, A.V.: Applied Functional Analysis, 2nd edition. New York, Heidelberg, Berlin: Springer-Verlag 1981.

[3] Barnett, S.: Introduction to Mathematical Control Theory. Oxford: Clarendon Press 1975.

[4] Brammer, R.F.: Controllability in Linear Autonomous Systems with Positive Controllers. SIAM J. Control 10, 339-353 (1972).

[5] Curtain, R.F.; Pritchard, A.J.: Functional Analysis in Modern Applied Mathematics. London, New York, San Fransisco: Academic Press 1977.

[6] Curtain, R.F.; Pritchard, A.J.: Infinite Dimensional Linear Systems Theory. Berlin, Heidelberg, New York: Springer-Verlag 1978.

[7] Fujimoto, T.; Krause, U.: An Ergodic Property for Certain Classes of Nonlinear Positive Operators. Preprint 1985.

[8] Grabowski, P.: Contribution to Stability Theory of Nonlinear Distributed Networks. Preprint 1986.

[9] Greiner, G.: Zur Perron-Frobenius-Theorie stark stetiger Halbgruppen. Math. Z. 177, 401-423 (1981).

[10] Greiner, G.: Asymptotics in Linear Transport Theory. Semesterbericht Funktionalanalysis, Tübingen, Sommersemester 1982, 71-98. See also: Spectral Properties and Asymptotic Behavior of the Linear Transport Equation. Math. Z. 185, 167-177 (1984).

[11] Kato, T.: Perturbation Theory for Linear Operators, 2nd edition. Berlin, Heidelberg, New York: Springer-Verlag 1976.

[12] Korobov, V.I.; Rabakh, R.: Exact Controllability in Banach Spaces. Diff. Uravn. 15, No. 12, 2142-2150 (1979). Also: Differential Equations 15, 1531-1537 (1979).

[13] Korobov, V.I.; Son, N.K.: Controllability of Linear Systems in Banach Spaces with Restrictions on the Control II. Diff. Uravn. 16, No. 6, 1010-1022 (1980). Also: Differential Equations 16, 633-642 (1980).

[14] Krein, M.G.; Rutman, M.A.: Linear Operators Leaving Invariant a Cone in a Banach Space. Uspehi Mat. Nauk (N.S.) 3, No. 1 (23) (1948), 3-95. Also: Translations Series 1, Vol. 10, 199-326. Providence, Rhode Island: American Math. Society 1962.

[15] Lamperti, J.: Stochastic Processes. New York, Heidelberg, Berlin: Springer-Verlag 1977.

[16] Nagel, R.: One-parameter Semigroups of Positive Operators. Berlin, Heidelberg, New York, Tokyo: Springer-Verlag 1986.

[17] Salamon, D.: Kontrollierbarkeit und Beobachtbarkeit unendlichdimensionaler linearer Systeme mit Anwendungen auf partielle Differentialgleichungen. Diplomarbeit, Universität Bremen 1977.

[18] Saperstone, S.H.; Yorke, J.A.: Controllability of Linear Oscillatory Systems with Positive Controls. SIAM J. Control 9, 253-262 (1971).

[19] Schaefer, H.H.: Topological Vector Spaces, 3rd print. New York, Heidelberg, Berlin: Springer-Verlag 1971.

[20] Schaefer, H.H.: Banach Lattices and Positive Operators. Berlin, Heidelberg, New York: Springer-Verlag 1974.

[21] Schanbacher, T.: Starkstetige Halbgruppen in der Kontrolltheorie. Semesterbericht Funktionalanalysis, Tübingen, Wintersemester 1983/84, 73-82.

[22] Son, N.K.: Local Controllability of Linear Systems with Restrained Controls in Banach Space. Acta Math. Vietnamica 5, 78-87 (1980).

[23] Triggiani, R.: Controllability and Observability in Banach Space with Bounded Operators. SIAM J. Control 13 (1975), 462-491.

[24] Triggiani, R.: Extensions of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators. SIAM J. Control 14 (1976), 313-338.

[25] Yosida, K.: Functional Analysis, 6th edition. Berlin, Heidelberg, New York: Springer-Verlag 1980.

LEBENSLAUF

Am 2.9.1958 wurde ich, Tilman Schanbacher, als Sohn des Studienprofessors Rainald Schanbacher und seiner Ehefrau Hedwig, geb. Rieber, in Reutlingen geboren.
Von 1965 bis 1968 besuchte ich die Jos-Weiß-Schule in Reutlingen, danach das Johannes-Kepler-Gymnasium in Reutlingen, wo ich 1977 die Reifeprüfung bestand.
Im Wintersemester 1977/78 begann ich mit dem Studium der Physik an der Eberhard-Karls-Universität in Tübingen, wechselte nach einem Semester um auf Mathematik, legte im Oktober 1979 die Diplom-Vorprüfung und im September 1983 die Diplom-Prüfung in Mathematik ab.
In den Wintersemestern 1980/81, 1983/84 und dem Sommersemester 1984 war ich als Wissenschaftliche Hilfskraft am Mathematischen Institut der Universität Tübingen tätig. Seit dem 1.4.1985 bin ich Wissenschaftlicher Angestellter am Institut für Medizinische Biometrie der Universität Tübingen.
Im Wintersemester 1984/85 war ich Stipendiat nach dem Landesgraduiertenförderungsgesetz, in dessen Rahmen mir im Februar und März 1985 ein Aufenthalt an der Heriot-Watt University in Edinburgh und dem Control Theory Centre an der University of Warwick ermöglicht wurde. Als DAAD-Stipendiat konnte ich im März und April 1986 die Polnische Akademie der Wissenschaften in Warschau besuchen.

Meine akademischen Lehrer waren die Herren Professoren und Dozenten G. Betsch, S. Breitsprecher, U. Felgner, W. Felscher, H. Fischer, H. de Groote, K.-P. Hadeler, H. Heyer, H. Kaul, R. Nagel, H.H. Schaefer, U. Schlotterbeck, A. Schönhage, E. Siebert, H. Sohr, A. Wolf (Mathematik), M. Baumann, A. Fäßler, F. Gönnenwein, H. Krüger, F. Lenz, O. Lutz, G. Mertens, A. Schwenk (Physik).

Druck und Verarbeitung: Dissertations Druck Darmstadt, Tel. 0 61 51 / 21 3 53